Last time A Banach $\ast$-algebra and its relation to the Fourier transform. Recall for $f \in L^1(\mathbb{R}^n)$, 

$$\hat{\chi}_x(f) = \int_{\mathbb{R}^n} f(y) e^{ix \cdot y} d\lambda(y)$$

defines a character. These characters are homomorphisms from the Banach $\ast$-algebra $L^1$ to $\mathbb{C}$. We check that it is a homomorphism:

$$\hat{\chi}_x(f * g) = \int_{\mathbb{R}^n} f(y) g(z - y) d\lambda(y) e^{ix \cdot z} d\lambda(z)$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) g(z - y) e^{ix \cdot z} d\lambda(y) d\lambda(z)$$

(Fubini’s gives)$$= \int_{\mathbb{R}^n} f(y) e^{ix \cdot y} \int_{\mathbb{R}^n} g(z) e^{ix \cdot z} d\lambda(z) d\lambda(y)$$

$$= \int_{\mathbb{R}^n} f(y) e^{ix \cdot y} \hat{\chi}_x(g) d\lambda(y)$$

Thus, each $x \in \mathbb{R}^n$ defines a character $\hat{\chi}_x$ on $L^1(\mathbb{R}^n)$. This can be used to map $f \in L^1(\mathbb{R}^n)$ to $\hat{\cdot} : \mathbb{R}^n \rightarrow C$; $\hat{f}(x) = \hat{\chi}_x(f) = \int_{\mathbb{R}^n} f(y) e^{ix \cdot y} d\lambda(y)$. The properties of $\hat{\chi}_x(x \in \mathbb{R}^n)$ can be summarized as

$$\hat{f} \hat{g} = \hat{f} \hat{g} = \hat{f}, \quad (f \hat{g}) = \hat{f} \hat{g}.$$ 

Moreover, $\hat{f} \in C_0(\mathbb{R}^n)$. Define

$$F : L^1(\mathbb{R}^n) \rightarrow C_0(\mathbb{R}^n)$$

$$f \mapsto \hat{f}$$

$F$ is the Fourier transform on $\mathbb{R}^n$. It is invertible on $\text{range} F$ so $L^1(\mathbb{R}^n)$ is isomorphic to a subalgebra of $C_0(\mathbb{R}^n)$. However, it is but not boundedly invertible since it is not onto. If it were onto, then by the bounded inverse theorem the inverse Fourier transform would be continuous (bounded) as a map from $C_0$ to $L^1$ and this isn’t true. For a counterexample, consider the function $f(x) = \frac{\sin x}{x} e^{-r|x|}$ on $\mathbb{R}$, as $r \rightarrow 0$, the Fourier transform converges to the characteristic function of a set while the sup norm stays bounded. However, the $L^1$-norm of the function goes to infinity which means that the inverse is unbounded.

We will also see later that the set $\{\hat{\chi}_x\}_{x \in \mathbb{R}^n}$ exhausts all characters.

Warm up: (Mystery of the missing unit)
1.47 Question. What if a Banach ∗-algebra does not have a unit?

Let \( \mathcal{A} \) be a Banach algebra and let \( \tilde{A} = \mathcal{A} \times \mathbb{C} \). Identify this Cartesian product as \( \mathcal{A} \times \mathbb{C} \), \((a, \lambda) = (a, 0) + (0, \lambda)\) equipped with multiplication \((a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)\), and norm \( \|(a, \lambda)\| = \|a\| + |\lambda| \). Then \( \tilde{A} \) is a Banach algebra with unit \((0, 1)\) and \( \mathcal{A} \) embeds in \( \tilde{A} \), \( \mathcal{A} \cong (\mathcal{A}, 0) \leq \tilde{A} \) as a subalgebra.

Note that if \( \mathcal{A} \) is ∗-algebra also then \((a, \lambda)^* = (a^*, \bar{\lambda})\) defines an involution on \( \mathcal{A} \).

We confirm associativity:

\[(a, \lambda)(b, \mu) = (a, \lambda)(bc + \mu c + \gamma b, \gamma \mu) = (a, \lambda)((b, \mu)(c, \gamma)).\]

Also, for all \((a, \lambda) \in \tilde{A}\),
\[(0, 1)(a, \lambda) = (a, \lambda) = (a, \lambda)(0, 1)\).

We check sub-multiplicativity next, for all \((a, \lambda), (b, \mu) \in \tilde{A}\),
\[\|(a, \lambda)(b, \mu)\| = \|(ab + \lambda b + \mu a, \lambda \mu)\| = \|ab + \lambda b + \mu a\| + |\lambda \mu| \leq \|a\| \|b\| + |\lambda| \|b\| + |\mu| \|a\| + |\lambda \mu| \leq \|(a, \lambda)\| \|(b, \mu)\| \leq \|(a, \lambda)(b, \mu)\|.

Finally, we check completeness: let \((a_n, \lambda_n)\) be a Cauchy sequence in \( \tilde{A} \), then there existsas \( N \in \mathbb{N} \) such that \( \|(a_n, \lambda_n) - (a_m, \lambda_m)\| \to 0 \) for all \( n, m > N \). Using the norm defined, we get that the sequences \( \|a_n - a_m\| \to 0 \) and \( |\lambda_n - \lambda_m| \to 0 \) for all \( n, m > N \) giving us Cauchy sequences \((a_n) \subset \mathcal{A}\) and \((\lambda_n) \subset \mathbb{C}\). By the completeness of \( \mathcal{A} \) and \( \mathbb{C} \), we have \( a_n \in \mathcal{A} \) and \( \lambda_n \in \mathbb{C} \) such that \((a_n) \to a\) and \((\lambda_n) \to \lambda\). Again, using the norm defined, we have that \( \|a_n - a\| + |\lambda_n - \lambda| \to 0 \) and thus \((a_n, \lambda_n) \to (a, \lambda)\) in \( \tilde{A} \). Using these properties we can proceed with Banach algebras even without a unit.

However, let us consider the case when \( \mathcal{A} \) has a unit, \( 1_{\mathcal{A}} \), and we embed it into \( \tilde{A} \). Due to uniqueness of the unit, \((0, 1)\) is the only unit in \( \tilde{A} \). Consider the element \((1_{\mathcal{A}}, 0)\) in \( \tilde{A} \), for any \((a, \lambda) \in \tilde{A}\),
\[(1_{\mathcal{A}}, 0)(a, \lambda) = (a + \lambda, 0)\]
and so we see that multiplication by \((1_{\mathcal{A}}, 0)\) gives us a projection onto \( \mathcal{A} \).

The missing unit: If \( \mathcal{A} \) has a unit, we take \( \tilde{A} = \mathcal{A} \); if not we take \( \tilde{A} \) as above.

1.48 Definition (Spectrum). For a Banach algebra \( \mathcal{A} \), \( a \in \mathcal{A} \) we call
\[\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \not\in G(\tilde{A})\}\]
the spectrum of \( a \) and \( \rho(a) = \mathbb{C} \setminus (\sigma(a)) \) the resolvent of \( a \).
1.49 Definition (Spectral radius). The number \( r(a) = \inf \{ r > 0 : \sigma(a) \subset B_r(0) \} \) is called the spectral radius of \( a \).

Note: If \( A \) does not have a unit and we adjoin 1 to it, then \( \|1\| = 1 \). We want to achieve this if \( A \) has a unit, if necessary by moving to an equivalent norm.

1.50 Theorem. Let \( A \) be a Banach algebra with unit 1, then there exists a norm \( \| \cdot \|_0 \) that is equivalent to \( \| \cdot \| \) (the norm on \( A \)) and \( \|1\|_0 = 1 \), and for \( a, b \in A \), \( \|ab\|_0 \leq \|a\|_0 \|b\|_0 \).

Proof. Consider for \( a \in A \), the map \( L_a : A \to A \), \( x \mapsto ax \) and let

\[
\|a\|_0 := \|L_a\| = \sup_{\|x\| \leq 1} ax \leq \|a\|.
\]

The map

\[
L : A \to \mathcal{B}(A)
\]

\( a \mapsto L_a \)

is linear: \( \forall x, L(a + b)(x) = L_{a+b}(x) = (a + b)(x) = ax + bx = L_a x + L_b x \implies L(a + b) = L_a + L_b \). Also, it is one-one: since \( L_a 1 = a \), if \( L_a = L_b \) then \( \forall x, ax = bx \) and for \( x = 1 \) we get \( a = b \). Thus the norm \( \| \cdot \|_0 \) norm given by \( \|a\|_0 := \|L_a\| \) is a well-defined norm on \( A \). Also,

\[
\|1\|_0 = \|1\| = \|\text{id}_A\| = \sup_{\|x\| \leq 1} x = 1
\]

and note that for all \( x \), \( L_{ab}(x) = (ab)x = a(bx) = L_a L_b(x) \) hence \( L_{ab} = L_a L_b \). Thus

\[
\|ab\|_0 = \|L_{ab}\|
\]

\[
= \|L_a L_b\|
\]

\[
\leq \|L_a\| \|L_b\|
\]

since the norm on \( \|L_a\| \) is the operator norm from \( \mathcal{B}(A) \) that is sub-multiplicative. Hence \( \|ab\|_0 \leq \|a\|_0 \|b\|_0 \). Next class, we show the equivalence of these norms.