Last time

- Banach algebra with and without a unit
- We had proved most of the following:

2.2 Theorem. Let $A$ be a Banach algebra with unit $1$, then there exists a norm $\| . \|_0$ that is equivalent to the norm on $A$ and satisfies $\| 1 \|_0 = 1$, and for each $a, b \in A$

$$\| ab \|_0 \leq \| a \|_0 \| b \|_0$$

Proof. We had $\| a \|_0 = \| L_a \| = \sup_{\| x \| \leq 1} \| ax \| \leq \| a \|$. It remains to show that $\| . \|$ and $\| . \|_0$ are equivalent. To see this, note

$$\| a \| = \| a 1 \| \leq \| L_a \| \| 1 \| = \| a \|_0 \| 1 \| \leq \| a \| \| 1 \|$$

so

$$\frac{1}{\| 1 \|} \| a \| \leq \| a \|_0 \leq \| a \|$$

From now on, we assume that if $1$ is a unit in a Banach algebra, then we can assume $\| 1 \| = 1$. Next, we study C*-algebras where $\| a \| = \| L_a \|$ for each $a \in A$.

2.3 Theorem. Let $A$ be a C*-algebra, then

1. If $L_a : x \mapsto ax$ as above, then $\| a \| = \| L_a \|$. In particular, if $1$ is a unit, then $\| 1 \| = 1$

2. If $A$ does not have a unit, then $\tilde{A}$ becomes a C*-algebra if we define $(a, \lambda)^* = (a^*, \bar{\lambda})$, and we choose the norm $\| (a, \lambda) \| := \| L_{(a, \lambda)} \|$ for $x \in A$

$$L_{(a, \lambda)} x = ax + \lambda x$$
Proof. (1) We have
\[ \|L_a\| = \sup_{\|x\| \leq 1} \|L_a x\| = \sup_{\|x\| \leq 1} \|ax\| \leq \|a\| \]
On the other hand, \( \|a a^*\| = \|a\|^2 = \|a\| \|a^*\| \), so if \( a = 0 \), nothing to show. Suppose \( a \neq 0 \), we let \( x = \frac{a^*}{\|a\|} \) and consider
\[
\|L_a\| = \sup_{\|x\| \leq 1} \|L_a x\| \geq \|L_a \frac{a^*}{\|a\|}\|
= \|a\| \frac{a^*}{\|a\|} = \|a\|
\]
We conclude \( \|a\| = \|L_a\| \). From 1 being a unit, \( L_1 = \text{id}_A \), and \( \|1\| = \|L_1\| = 1 \).

(2) Let \( L : \tilde{A} \rightarrow B(A) \) given by \( L(a, \lambda) = L_{(a, \lambda)}, \quad L_{(a, \lambda)} x = ax + \lambda x \). We omitted the proof that \( \tilde{A} \) is a Banach space. We will only show the norm property of a C*-algebra.

First, we show \( L \) is 1-1. Let \( L(a, \lambda) = 0 \). If \( \lambda = 0 \), then \( L_a = 0 \) so \( a = 0 \). Suppose \( \lambda \neq 0 \), then by linearity,
\[
0 = L_{(a, \lambda)} x = ax + \lambda x \implies (-\frac{1}{\lambda})ax - x = 0
\]
implies that \( (-\frac{1}{\lambda})a \) is a (left) unit in \( A \) which contradicts our assumption that \( A \) does not have a unit. Thus \( L \) is 1-1, and \( \|(a, \lambda)\| = \|L(a, \lambda)\| \) is a norm which extends the norm on \( A \).

To check the norm property, we only need to show for \( x \in \tilde{A}, \quad \|x\|^2 \leq \|x^* x\| \).
If \( \|x\| = 0 \), nothing to show.
If \( 0 < r < \|x\| \), by definition of norm on \( \tilde{A} \) and the result above, we have
\[
r < \|x\| = \|L_x\| = \sup_{\|y\| \leq 1} \|L_x y\| = \sup_{\|y\| \leq 1} \|xy\|
\]
so there is \( y \in A, \quad \|L_x y\| = \|xy\| \geq r \).
Using the submultiplicativity property of \( \tilde{A} \), consider \( y \equiv (y, 0) \), we have
\[
\|x^* x\| \geq \|y\| \|y^*\| \|x^* x\| \|y\| \geq \|y^* x^* yx\| = \|(xy)^* xy\| \quad \forall y \in A, C^*-algebra
\]
so taking the sup over all \( r < \|x\| \) gives \( \|x^* x\| \geq \|x^2\| \)
\[
\]

We conclude with examples

2.4 Example. Let \( X \) be a locally compact Hausdorff space that is not compact, let \( A = C_0(X) \), then \( A \) does not have a unit and \( \tilde{A} \) can be thought of continuous functions with limit at infinity, with \( (0, 1) \equiv 1 \). To justify this, we note \( \tilde{A} \rightarrow C_b(X), \ (f, \lambda) \mapsto f + \lambda 1 \) can be thought if as an isometry, where \( C_b(X) \) has the sup-norm.

2.5 Remark. \( C_0(X) \) does not have a unit since the constant function \( 1 \) is not included in \( A \) as it does not go to 0 at infinity.
Proof. We want to identify \((f, \lambda) = f + \lambda \mathbb{I}\).

If we choose \(\|f, \lambda\| = \|f\|_{\infty} + |\lambda|\), then \(\|f + \lambda \mathbb{I}\|_{\infty} \leq \|f\|_{\infty} + |\lambda|\) but the equality may not hold.

If instead, we choose \(\|f, \lambda\| = \|L(f, \lambda)\|\), then Urysohn’s lemma guarantees the existence of a function \(g = (f, \lambda) \in \tilde{A}\) such that \(g(x) = 1\) for \(x \in K=\text{compact}\), and \(g(x) = 0\) where \(x \not\in K\).

\[
\|L(f, \lambda)\| = \sup_{\|x\| \leq 1} |f(x) + \lambda x| = \|f + \lambda \mathbb{I}\|_{\infty}
\]

Hence \(\tilde{C}_{0}(X)\) is a closed subspace of \(C_{b}(X)\) which is isometrically embedded in \(C_{b}(X)\) as a space of continuous functions that have limit at infinity. \(\square\)

2.6 Example. Let \(X\) be compact Hausdorff space \(A = C(X)\), \(f \in A\). What is \(\sigma(f)\)?

We recall \(g \in G(A)\) means there exists \(h \in C(X)\) and \(gh = 1\) so \(g(x) \neq 0\) for each \(x \in X\). Conversely, if \(g(x) \neq 0\) for each \(x \in X\), then \(h = \frac{1}{g(x)}\) is in \(C(X)\).

Next, to see what the spectrum is, note \(f - \lambda \mathbb{I}\) is invertible if and only if \(f(x) \neq \lambda\) at any \(x \in X\). Consequently, \(\sigma(f) = \{\lambda : f(x) = \lambda\) for some \(x \in X\} = f(X)\).

2.7 Example. Let \(X\) be a locally compact Hausdorff space \(A = C_{0}(X)\). What is \(\sigma(f)\)?

For \(f \in \tilde{A}\), then \(f \in C(X)\) and \(f\) has limit at infinity. So if \(f\) is invertible, then \(f(x) \neq 0\) for each \(x\), and \(\lim_{x \to \infty} f(x) \neq 0\). Otherwise, taking \(1/f\) would diverges at infinity, so not give a function in \(\tilde{A}\). In notation of \(\tilde{A}\), \((f, \lambda)\) is invertible iff \(f(x) \neq 0\) for each \(x\) and \(\lambda \neq 0\). Hence \(\sigma(f) = f(X) \cup \{0\}\).

2.8 Example. Let \(A \subset B(\mathbb{C}^{n})\) be an algebra of \(n \times n\) matrices containing \(\mathbb{I}\). For \(a \in A\), what is \(\sigma(a)\)?

Let \(x \in A\) be invertible in \(A\) if and only if there is \(y \in A\) s.t \(xy = yx = \mathbb{I}\). We show that if there is \(y \in B(\mathbb{C}^{n})\) with \(xy = yx = \mathbb{I}\) then \(y^{-1} \in A\). To see this, note \(L_{x} : A \to A\), so if \(x\) is invertible in \(B(\mathbb{C}^{n})\), then the map \(y \mapsto xy\) is 1-1. Since \(A\) is finite dimensional, \(L_{x}\) is also onto. So there exists \(y \in A\) s.t \(L_{x}(y) = xy = \mathbb{I}\). Hence \(x^{-1} = y \in A\).

Thus \(G(A) = A \cap G(B(\mathbb{C}^{n})) = \{a \in A : \det a \neq 0\}\).

From this, we deduce for \(a \in A\), \(\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \mathbb{I} \notin G(A)\}\). Hence the spectrum consists of eigenvalues of \(a\).

2.9 Remark. It is interesting to note \(\sigma(a)\) does not depend on the choice of \(A \subset B(\mathbb{C}^{n})\).