Last time

- Banach space clean up.
- Properties of the spectrum.

1.0.1 Warm-up (review)

We start by reviewing the Hahn-Banach theorem, a powerful theorem which allows us to easily compute with vectors via their “coordinates” (see first-day handout).

1.6 Theorem (Hahn-Banach, complex version). Given a complex normed space $X$, and a subspace $Y$ with a bounded linear functional $f : Y \to \mathbb{C}$ such that $\|f\| = M$, there exists a linear functional $g : X \to \mathbb{C}$ such that $g|_Y = f$ and $\|g\| = M$.

1.7 Remark. Note that although Banach is in the name of the theorem, it has nothing to do with Banach spaces or Banach algebras. In fact $X$ need not be a normed space; it may have only a seminorm, or, simply a function $p : X \to \mathbb{R}$ satisfying

- $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$, and
- $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}^+$.

An important consequence is that the set of bounded linear functionals on a Banach algebra distinguishes between any two elements in the space.

1.8 Corollary. If $A$ is a Banach algebra and $a, b \in A$, $a \neq b$, then there exists a bounded linear functional $g : A \to \mathbb{C}$ such that $g(a) \neq g(b)$.

Proof. Consider the subspace generated by $a - b$, $Y = \mathbb{C}(a - b)$. Let $f : Y \to \mathbb{C}$ be linear with $f(a - b) = \|a - b\|$. So we have

$$\|f\| = \sup_{\|a-b\|\leq 1} |f(a - b)| = \sup_{\|a-b\|\leq 1} \|a - b\| = 1.$$ By Hahn-Banach theorem, we know that there exists $g : X \to \mathbb{C}$ such that $g(a - b) = \|a - b\| \neq 0 \Rightarrow g(a) \neq g(b)$.
The next important result is a consequence of the Baire category theorem. Note that in the following, we can use the terms 'nonmeager' or of 'second-category' to describe a subset $B \subset X$ that is not a countable union of nowhere dense sets of $X$.

1.9 Theorem (Uniform boundedness or Banach-Steinhaus Theorem). Let $X$ be a Banach space, $B$ a set that is not a countable union of nowhere dense sets in $X$, and $\Gamma$ a collection of bounded linear functionals. Suppose for any $x \in B$, \{\lambda x : \lambda \in \Gamma\} is bounded. Then $T$ is uniformly bounded, namely,

$$\sup_{\lambda \in \Gamma} \|\lambda\| < \infty.$$  

We prove the following special case.

1.10 Corollary. If $A$ is a Banach algebra, and $(a_n)_{n \in \mathbb{N}}$ a sequence such that for any $f \in A'$, $(f(a_n))_{n \in \mathbb{N}}$ is bounded, then

$$\sup_{n \in \mathbb{N}} \|a_n\| < \infty.$$  

- This is another example of using a property that holds for the coordinates to show it holds for the entire space.
- Here we are saying that "weak boundedness" of the sequence implies boundedness of the sequence; this does not hold when we are talking about convergence because weak convergence does not imply strong.

Proof. By Hahn-Banach, we know that for each $a \in A$, there exists an $f \in A'$ such that $f(a) = \|a\|$ and $\|f\| = 1$. We can see how $a$ acts on the $f$'s by writing

$$\|a\| = \sup_{\|f\| \leq 1} |f(a)|.$$  

In particular if we define

$$i : A \to (A')', \quad a \mapsto \{f \mapsto f(a)\},$$

then we have

$$\|i(a)\| = \|a\|,$$

so $i$ is an isometry. To see that the equality holds, note that the canonical embedding $i$ maps $a$ to a bounded linear functional $i(a)(\cdot)$ on $A'$, which then takes a bounded linear functional $f$ on $A$ to its point evaluation $i(a)(f) = f(a) \in \mathbb{C}$. Now when we write the norm

$$\|i(a)\| = \sup_{\|f\| \leq 1} |i(a)(f)| = \sup_{\|f\| \leq 1} |f(a)|,$$

this is the equality we have coming from Hahn-Banach.

Moreover, if we assume that $(f(a_n))_{n \in \mathbb{N}} = (i(a_n)(f))_{n \in \mathbb{N}}$ stays bounded for each $f \in A'$, then since $A'$ is a Banach space, by uniform boundedness we have

$$\sup_{n \in \mathbb{N}} \|i(a_n)\| = \sup_{n \in \mathbb{N}} \|a_n\| < \infty.$$  

$\square$
1.0.2 Back to proof from last time

We were proving the following theorem:

1.11 Theorem. Let $A$ be a Banach algebra with unit 1 and $\|1\| = 1$. We have:

(i) For $\|x\| < 1$, the element $1 - x$ is invertible, and

\[
(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n.
\]

Moreover,

\[
\|(1 - x)^{-1}\| \leq \frac{1}{1 - \|x\|}
\]

and

\[
\|(1 - x)^{-1} - 1\| \leq \frac{\|x\|}{1 - \|x\|}.
\]

(ii) $G(A)$ is an open subset of $A$. More precisely, for each $a \in G(A)$, if $r = \|a^{-1}\|^{-1}$, then

$B_r(a) \subset G(A)$. Also, $G(A)$ is a group whose operations (multiplication and inversion) are continuous.

(iii) For each $a \in A$, $\sigma(a)$ is a compact subset of $\mathbb{C}$ and $\tau(a) \leq \|a\|$.

(iv) The function

\[
R : \rho(A) \to A, \quad \lambda \mapsto (a - \lambda 1)^{-1},
\]

is continuous and even analytic, that is, for each $a \in A$, $\lambda_0 \in \rho(a)$, there is $\tau > 0$ such that for $\lambda \in B_\tau(\lambda_0)$,

\[
R(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda_0)(\lambda - \lambda_0)^n,
\]

with some sequence $(a_n(\lambda_0))$ in $A$ such that the series converges in the norm of $A$.

Proof of (ii). Let $a \in G(A)$, and assume $\|x\| < \|a^{-1}\|^{-1}$. Then $a - x = a(1 - a^{-1}x)$ and thus

\[
\|a^{-1}x\| \leq \|a^{-1}\|\|x\| < 1.
\]

From (i), the element $1 - a^{-1}x \in G(A)$, and since $a \in G(A)$, we have

\[
a(1 - a^{-1}x) = a - x \in G(A).
\]

This works for any such $x$, so the claim follows. To see this, let $y \in B_r(a) \equiv \{z \in A : \|z - a\| < r\}$ with $r = \|a^{-1}\|^{-1}$. Then defining $x = a - y$ gives

\[
\|x\| < \|a^{-1}\|^{-1},
\]

and thus

\[
y = a - x \in G(A).
\]
To understand that $G(A)$ is a topological group, we need to show continuity of the group operations: the product and the inversion. We already showed continuity of the product in $A$, hence this also holds for the subset $G(A)$. We need to show continuity of the inversion.

To this end, let $a \in G(A)$ and $\|x\| < \|a^{-1}\|^{-1}$. Then $a - x \in G(A)$ and

$$(a - x)^{-1} = (1 - a^{-1}x)a^{-1}.$$ 

We know that the right-hand side is (sequentially) continuous at $x = 0$, and so is the left-hand side. Thus $a \mapsto a^{-1}$ is a continuous map at each $a \in G(A)$.

**Proof of (iii).** We know that $R : \rho(a) \to A$, $\lambda \mapsto (a - \lambda1)^{-1}$ is continuous at $\lambda \in \rho(a)$ by (ii), so $\rho(a)$ is open and hence $\sigma(a)$ is closed. For $|\lambda| > \|a\|$, we get

$$\lambda1 - a = \lambda(1 - \lambda^{-1}a)$$

and thus

$$\|\lambda^{-1}a\| = \frac{\|a\|}{|\lambda|} < 1.$$ 

From (i), we have

$$1 - \lambda^{-1}a \in G(A),$$

so also

$$\lambda(1 - \lambda^{-1}a) = \lambda1 - a \in G(A),$$

meaning $\lambda \in \rho(a)$. We have shown that $\lambda \in \mathbb{C}$ with the property $|\lambda| > \|a\|$ implies $\lambda \in \rho(a)$; in symbols,

$$\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\} \subset \rho(a).$$

Then taking complements we have

$$\sigma(a) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\},$$

and this implies

$$r(a) \equiv \sup_{\lambda \in \sigma(a)} |\lambda| \leq \|a\|.$$ 

Hence we see that $\sigma(a)$ is bounded in $\mathbb{C}$. By closedness, $\sigma(a)$ is compact.

**Proof of (iv).** Let $\lambda_0 \in \rho(a)$, $r = \|R(\lambda_0)\|^{-1}$. For $|\lambda - \lambda_0| < r$, we get $\lambda \in \rho(A)$ from (ii). The proof of (ii) gives an expression for $R(\lambda)$

$$R(\lambda) = (\lambda1 - a)^{-1}$$

$$= (\underbrace{(\lambda_01 - a)}_{A} - \underbrace{(\lambda_0 - \lambda)}_{X})^{-1}$$

$$= \underbrace{(1 - R(\lambda_0)(\lambda_0 - \lambda))^{-1}}_{(1-A^{-1}X)^{-1}} R(\lambda_0).$$

4
(Note we use the uppercase A to highlight the similarities between what we have here and when we considered \((a - x)^{-1}\). Not to be confused with the name of the Banach algebra.) We obtain the Neumann series

\[
\begin{align*}
R(\lambda) &= \left( \sum_{n=0}^{\infty} R(\lambda_0)^n (\lambda_0 - \lambda) \right) R(\lambda_0) \\
&= \sum_{n=0}^{\infty} R(\lambda_0)^{n+1} (-1)^n (\lambda - \lambda_0)^n
\end{align*}
\]

and the series converges using

\[
\|R(\lambda_0)(\lambda - \lambda_0)\| = |\lambda - \lambda_0|\|R(\lambda_0)\| < 1.
\]

So, we have shown (iv).

It remains to show that the spectrum is nonempty to finish elementary properties.