0 Warm-Up

Recall that \(0 \leq a_{n+m} \leq a_n a_m\) implies \(a_n^{1/n} \to \inf a_n^{1/n}\) for \(a_n \in \mathbb{R}\). Looking at the edge case of \(a_{n+m} = a_n a_m\), we see that \(a : \mathbb{N} \to \mathbb{R}^+\) is a semigroup homomorphism determined by \(a_n = a_1^n\).

1 The Spectral Theorem

1.0.1 Theorem. Let \(a \in A, A\) a Banach Algebra. Then we have the following results for the spectrum:

(i) \(\sigma(a) \neq \emptyset\)

(ii) \(r(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n}\)

Proof. Assume \(\sigma(a) = \emptyset\) Then \(\rho(a) = \mathbb{C} - \sigma(a) = \mathbb{C}\) \(R : \mathbb{C} \to A\) s.t. \(R(\lambda) = (\lambda 1 - a)^{-1}\) is analytic in \(\mathbb{C}\), so for \(\lambda > \|a\|\) we get \(\|R(\lambda)\| = \|(\lambda 1 - a)^{-1}\| = \|\lambda\|(1 - \lambda^{-1}a)\| = \lambda^{-1}\|1 - \lambda^{-1}a\| < \lambda^{-1}\frac{1}{\|a\|}\)

Hence \(R\) is bounded on \(\{\lambda \in \mathbb{C} : \|\lambda\| \geq \|a\|\}\). On the other hand, \(R\) is continuous on any closed disk in \(\mathbb{C}\), so \(R\) is bounded on all of \(\mathbb{C}\). Now let \(f \in A'\) (the dual of \(A\)). Then \(f \circ R : \mathbb{C} \to \mathbb{C}\) is bounded analytic, hence it is constant so \(f \circ R(\lambda) = f \circ R(0) = f(-a)\)

Since \(\{f : f \in A'\}\) distinguishes between any 2 elements in \(A\), we have \(R(\lambda) = -a^{-1}\). But then \(\lambda 1 - a = R(\lambda)^{-1} = (-a)^{-1} = -a\) for each \(\lambda \in \mathbb{C}\) for each \(\lambda \in \mathbb{C}\). Contradiction! This proves (i)

Now we prove item (ii). Let \(s(a) = \inf_n \|a^n\|^{1/n}\), then as in the warm-up, we have \(s(a) = \lim_{n \to \infty} \|a_n\|^{1/n}\). We show if \(\lambda|s(a)\), the \(\lambda \in \rho(a)\). Note that \(\limsup (\|\lambda^{-1}a\|^{1/n}) = \lim_{n \to \infty} \|a\|^{1/n} < 1\). Therefore by Asymptotic bound, \(\sum_{n=0}^{\infty} \lambda^{-1}a^n\) converges.

Recall that if \((\lambda 1 - a)^{-1} = \frac{1}{\lambda}(1 - \lambda^{-1}a)^{-1}\) we have \(R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1}a^n\) thus \(\lambda \in \rho(a)\). By comparing \(\sigma(a)\) and \(\rho(a)\) we see that \(r(a) = \sup(\{\lambda : \lambda \in \sigma(a)\}) \leq s(a)\). To show equality between \(s(a), r(a)\) we let \(r > r(a)\), \(\Omega = \{z \in \mathbb{C} : |z| > r(a)\} \subset \rho(a)\). For each \(f \in A'\), we have \(f \circ R : \Omega \to \mathbb{C}\) is analytic and has the series expansion \(f \circ R(z) = \sum_{n \in \mathbb{Z}} c_n z^n\), but for \(|z| > \|a\|\) we
know $R(z) = \sum_{n=0}^{\infty} z^{-n-1} a^n$. Comparing coefficients give $c_n = 0$ for $n \geq 0$ and $c_{-1-n} = f(a^n)$ for $n \geq 0$. By our choice of $z$, $\lim_{n \to \infty} f(a^n)z^{-n} = 0$. Then by Banach-Steinhaus, we see that for $z \in \Omega$, $(z^{-n}a^n)_{n \in \mathbb{N}}$ is uniformly bounded. Therefore there exists $C > 0$ with $\|a^n\| \leq C|z|^n$ for all $n \in \mathbb{N}$. Thus $s(a) \leq \lim_{n \to \infty} C^{1/n}|z| = |z|$. This works for each $z$ with $|z| > r(a)$. Taking the infimum over all such $z$ yields $s(a) \leq r(a)$. Thus $s(a) = r(a)$. 

1.0.2 Remark. It is important to observe that the identity $r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$ relates algebraic and topological quantities without assuming $C^*$ algebra structure.

2 Consequences of the Theory of the Spectrum

We explore the consequences of spectral theory.

2.0.1 Theorem. Let $A$ be a Banach Algebra with unit 1, in which each element $a \neq 0$ is invertible, then $A \cong \mathbb{C}$ and $\dim = 1$.

Proof. Let $a \in A$. By the above theorem on the spectrum, the exists $\lambda \in \sigma(A)$ s.t. $\lambda 1 - a \notin G(A)$ but by assumption $\lambda 1 - a = 0$ thus $a = \lambda 1$.

In $C^*$ algebras, we find more direct relationships between the spectrum and the norm.

2.0.2 Lemma. Let $A$ be a $C^*$ algebra.

(i) $\sigma(a^*) = \overline{\sigma(a)}$

(ii) if $a$ is normal, $r(a) = \|a\|$

(iii) for $a \in A$, $\|a\| = \sqrt{r(a^*a)}$

Proof. i) We know that $a - \lambda 1$ is invertible in $\tilde{A}$ iff $(a^* - \lambda 1)^* = a^* - \overline{\lambda}1$. This gives the relation.

The rest of the proof is to be continued....