Last Time

- Spectral radius is norm in \( \mathbb{C}^* \)- algebra.
- Properties of the spectrum under homomorphism, with or without respecting involution.

Finishing the theorem from last time-

\[ \text{2.51 Theorem.} \quad \text{Let } A \text{ be a Banach *-algebra and } B \text{ a } \mathbb{C}^*\text{-algebra and } f : A \rightarrow B \text{ is a homomorphism (} f \text{ is an algebraic homomorphism, is bounded and } f(a^*) = (f(A))^* \text{ for each } a \in A). \text{ then } f \text{ is a contraction. i.e., } \|f(a)\| \leq \|a\| \text{ for all } a \in A. \]

**Proof.** WLOG, let us suppose \( B = \overline{f(A)} \), we observed \( f(1) = 1 \), or if \( A \) does not have unit we apply this to \( \tilde{A}, \tilde{B} \) and achieve \( \tilde{f} \) extends to \( \tilde{f} \) with \( f(1) = 1 \). Consider \( a = a^* \in A \), then \( f(a) \) is normal(since \( f(a)(F(a))^* = f(a)f(a^*) = f(aa^*) = f(a^*)f(a) = f(a^*)f(a) = (f(a))^*f(a) \) as \( f \) is homomorphism). Now using spectral radius properties, we have

\[
\|f(a)\| = r(f(a)) \overset{\text{Lemma}}{\leq} r(a) \leq \|a\|
\]

For genral \( a \in A \), we have

\[
\|f(a)\|^2 = \|f(a^*)f(a)\| \\
= \|f(a^*a)\| \\
\leq \|a^*a\| \\
\leq \|a^*\|\|a\| \\
= \|a\|^2
\]

using subadditivity of norm. Therefore we have \( \|f(a)\| \leq \|a\| \) for any \( a \in A \). \( \square \)

**Warm up:** Without completeness of \( A \) and boundedness of \( f \), the conclusion about the set of homomorphism does not hold. Consider the Banach-*-algebra \( l^1(\mathbb{N}) \) and \( A = C_\infty(\text{Sequence with finitely many non-zero elements}) \) in \( l^1(\mathbb{N}) \), then \( C_\infty \) is a subalgebra of \( l^1(\mathbb{N}) \) with norm and involution(identity).
For each $z \in \mathbb{C}$, we can define $f : A \to \mathbb{C}$ by $x \mapsto \sum_{n=1}^{\infty} x_n z^n$ (where the sum is well defined as only finitely any $x_n$'s are non-zero).

By $A$ dense in $l^1(N)$ and $(l^1(N))' = l^\infty(N)$, we have

$$
\|f\|_1 = \sup_{\|x\| \leq 1} |f(x)|
\leq \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |x_n| |z|^n
\leq \sup_{\|x\| \leq 1} \|x\| \sup_{n \in \mathbb{N}} |z|^n
\leq \sup_{n \in \mathbb{N}} |z|^n
$$

Moreover for $x = e_1$, a canonical basis element of $l^1(N)$ and $|z| < 1$,

$$
|f(e_1)| = |z| = \sup_{n \in \mathbb{N}} |z|^n
$$

Hence, we have $\|f\| = \sup_{n \in \mathbb{N}} |z|^n$. Also if $|z| > 1$, then $\|f\| = \infty$.

**Note.** We have a homomorphism which is everywhere defined but is discontinuous.

If $B = \mathbb{C}$, we can make a similar statement as in above theorem, but we do not require the presence of involution.

**2.52 Theorem.** Let $A$ be a Banach algebra, $f : A \to \mathbb{C}$ a homomorphism (i.e., in $A'$), then $\|f\| \leq 1$. If $A$ has a unit and $f \neq 0$, then $f(1) = 1$, so $\|f\| = 1$.

**Proof.** Let us assume $\|f\| > 1$, then there is $a \in A$ with $\|a\| = 1$ and $|f(a)| > 1$ (using the definition of the operator norm), so by scaling we also get $\|a\| < 1$ and $f(a) = 1$ (as $\|f(a)\| = \|a\| < 1$ and $f(\frac{a}{\|f(a)\|}) = 1$). Now $A$ being a Banach algebra, so by complexity of $A$,

$$
b = \sum_{n=1}^{\infty} a^n \in A
$$

where RHS converges (in norm) as $\|a\| < 1$. And by examining of the power series, we see

$$
a + ab = a + a(\sum_{n=1}^{\infty} a^n) = a + \sum_{n=2}^{\infty} a^n = b
$$

Applying $f$ gives,

$$
f(b) = f(a) + f(a)f(b) = 1 + f(b)
$$

this is a contraction, so $\|f\| \leq 1$.

If $A$ has a unit $1$ and $f \neq 0$, then $f(1)^2 = f(1)$, so $f(1) = 1$ otherwise $f = 0$. Thus, $\|f\| = 1$, because $\|1\| = 1$. 

2
2.53 Definition. For a complex algebra $A$, we write $\Gamma_A$ for the space of all non-zero homomorphism $\chi : A \to \mathbb{C}$(If $A$ is Banach algebra, these elements of $\Gamma_A$ are continuous, hence contractive).

We then call $\Gamma_A$ the spectrum of $A$.

2.54 Remarks. 1. If $S$ is an involutive semigroup and $l^1(S)$ the Banach-$*$-algebra defined with $f^*(s) = f(s^*)$,

$$(f \ast g)(s) = \sum_{a,b \in S, ab = s} f(a)g(b),$$

then we had shown that continuous characters $\chi : l^1(S) \to \mathbb{C}$ come from bounded character on $S$ by

$$\chi(f) = \sum_{s \in S} f(s)\gamma(s)$$

with $\gamma(s) = \chi(\delta_s)$, Now we know that all of these are infact contractive characters.

2. Let $X$ be locally compact but not compact but not compact, $A = C_0(X)$. We show that each non-zero homomorphism $\chi : A \to \mathbb{C}$ is given by $\chi(f) = f(x)$ for $x \in X$.

To see this, we extend $\chi$ to $\tilde{A} \subset C_b(X)$, by $\chi(1) = 1$.

Let $N = \ker(\chi)$, then $N$ is a (closed) subspace of $A$ of co-dimension one(by Rank-Nullity) and $\tilde{A}N \subset N$ (since for $\tilde{a} \in \tilde{A}$, $x \in \ker(\chi)$, we have $\chi(\tilde{a}x) = \chi(\tilde{a})\chi(x) = \chi(\tilde{a})0 = 0$)