Functional Analysis, Math 7320
Lecture Notes from February 07, 2017

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Properties of reflexivity with respect to weak and weak-* topologies

We recall that for a TVS X, the weak topology on X is the coarsest topology that makes all elements in X' continuous on X. We write X_w for (X, τ_w).

Warm-Up:

By the fact that $\tau_w \subset \tau$, for any $A \subset X$ we have:

(i) A is w-open $\Longrightarrow$ A is open.

(ii) A is w-closed $\Longrightarrow$ A is closed.

2.37 Question. What is the relationship between weak compactness and compactness in a Hausdorff space?

2.38 Answer. A is w-compact $\Longleftarrow$ A is compact.
Indeed, assume A compact in X, τ, i.e. for any open cover $A \subset \underset{i \in I}{\cup} S_i$ where $S_i \in \tau$ we can find a finite sub-cover $A \subset \underset{j \in J}{\cup} S_j$ (J finite subset of I). Now, assume an open cover of A in (X, τ_w), $A \subset \underset{i \in I}{\cup} W_i$, where $W_i \in \tau_w$. Since $\tau_w \subset \tau$, $\{W_i\}_{i \in I}$ is also an open cover of A in (X, τ), thus exists a sub-cover of A from sets in $\{W_i\}_{i \in I}$.

2.39 Remark. Recall that if X is a locally TVS and C is a convex subset, then $\overline{C} = \overline{C}^w$. (last Theorem on 11/17/2016, proved on 11/22/2016)

Proof. Indeed By $\tau_w \subset \tau$ we know that $\overline{C} \subset \overline{C}^w$. Now assume $x \notin \overline{C}$. From a separation Theorem (version of Hahn-Banach, we also use it in the first proof on 11/22/2016), there is $f \in X'$ with $Ref(x) < s = infRef(\overline{C})$. By the (weak) continuity of f, $U = \{x \in X : Ref(x) < s\}$ is weakly open and disjoint from C, so also disjoint from $\overline{C}^w$. Hence, $x \notin \overline{C}^w$. We get

$$(\overline{C})^C \subset (\overline{C}^w)^C$$

so

$$\overline{C}^w \subset \overline{C}$$

We conclude that $\overline{C}^w = \overline{C}$. □

2.40 Remark. We also recall that: the weak-* topology on X' has the property that, if $g \in (X', w^*)$ then for each $f \in X'$ we have $g(f) = f(x)$ for some $x \in X$.(see definition of weak-* topology and the comments before this definition on 11/22/2016)

2.41 Corollary. *If X is not reflexive, then there exists a convex set C in X' such that $\overline{C} \neq \overline{C}^{w^*}$.*

Proof. Take $g \in X'' \setminus i(X)$. Note that such a g exists since X is not reflexive. Then $C = ker(g)$ is convex and (norm) closed. Assuming $\overline{C} = \overline{C}^{w^*}$ would give $C = \overline{C}^{w^*}$, which implies (by Theorem 11.6.10 on 9/29/2016)) that g is w^*-continuous and thus $g(f) = i(x)(f) = f(x)$ for some $x \in X$. This contradicts our choice of g. □

2.42 Remark. We also recall that: if X is a normed vector space then, by Banach-Alaoglu, the closed unit ball in X', $\overline{B_1}^{X'}$, is weak-* compact (first Theorem after our definition of weak-* topology on 11/22/2016). Moreover, if X is separable, then $\overline{B_1}^{X'}$ is w^*-sequentially compact (first Corollary on 11/29/2016, before Krein Milman section).

Because of the importance of this last result, we present another proof, more self-contained than the one we presented already.

2.43 Theorem. *(Helly) Let X be a separable Banach space. Then $\overline{B_1}^{X'}$ in X' is $w*$-sequentially compact.*

Proof. Let S be a countable dense set in X and $(g_n)_{n\in\mathbb{N}}$ in $\overline{B_1}^{X'}$. Then, iteratively passing to convergent subsequences, we get that $\lim_{k\to\infty} g_{n_k}(s)$ exists for each $s \in S$.
Next, for all $x \in X$, $(g_{n_k}(x))_{k\in\mathbb{N}}$ is Cauchy, because for all $\epsilon > 0$ there exists $s \in S$ with $||x - s|| < \frac{\epsilon}{3}$ and by convergence on S, exists $N \in \mathbb{N}$ for which, if $j, k \leq N$ we have

$$|g_{n_k}(s) - g_{n_j}(s)| < \frac{\epsilon}{3}$$

Using that $||g_{n_k}|| \leq 1$ and continuity estimates we get

$$|g_{n_k}(s) - g_{n_j}(s)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Next, defining $g(x) = \lim_{k\to\infty} g_{n_k}(x)$, the limit of the the sequence $g_{n_k}(x)$ of continuous linear maps for all $x \in X$, using Banach-Steinhaus and it's consequences (first Theorem and following Corollary on 11/01/2016) we get that $g \in X'$ and $||g||_{X'} \leq 1$.
We conclude that $g_{n_k} \longrightarrow g \in \overline{B_1}^{X'}$ in weak-* topology (by definition). □

2.44 Question. What if X is not separable?

2.45 Answer. Reflexivity is another sufficient condition that guarantees sequential compactness.

We prepare this result by considering $\overline{B_1}^{X''}$

2.46 Theorem. *(Goldstine) Let X be a normed space. Then*

$$\overline{B_1}^{X''} \subset \overline{i(\overline{B_1}^{X})}^{w^*}$$

Proof. By Banach-Alaoglu, $\overline{B_1}^{X''}$ is weak-* compact, so it is weak-* closed. Since i is an isometry

$$i(\overline{B_1}^X) \subset \overline{B_1}^{X''}$$

Taking the weak-* closure, since $\overline{B_1}^{X''}$ is already weak-* closed, we get

$$\overline{i(\overline{B_1}^X)}^{w^*} \subset \overline{B_1}^{X''}$$

Next, assume that there exists $y \in \overline{B_1}^{X''} \setminus \overline{i(\overline{B_1}^X)}^{w^*}$. By convexity and the separation Theorem for locally convex TVS (first Theorem on 11/15/2016 where we obtained the strict inequality), there exists a weak-* continuous linear functional g on X'' such that

$$Reg(y) < inf\{Reg(u) : u \in \overline{i(\overline{B_1}^X)}^{w^*}\}$$

By weak-* continuity, there exists some $z \in X'$ with $g = i_z$. Let, for some $u \in X''$, $f(u) = -i(z)(u)$. Then, for each $x \in X$, there exists $c \in \mathcal{K}$, $|c| = 1$, such that

$$|z(x)| = z(cx) = Rez(cx)$$

Using that $c\overline{B_1}^X = \overline{B_1}$ we get

$$\begin{aligned}
||f||||y|| &\geq |f(y)| \\
&\geq Ref(y) \\
&> sup\{Ref(u) : w \in \overline{i(\overline{B_1}^X)}^{w^*}\} \\
&\geq sup\{Ref(u) : u = i(x) \text{ for some } x \in \overline{B_1}^X\} \\
&= sup\{-Rez(x) : x \in \overline{B_1}^X\} \\
&= sup\{|z(x)| : x \in \overline{B_1}^X\} \\
&= ||i(z)|| = ||f|| \quad \text{(since i is an isometry)}
\end{aligned}$$

Thus, by assumption, $||y|| > 1$ or $y \notin \overline{B_1}^{X''}$. Hence, by inclusion of complements

$$\overline{B_1}^{X''} = \overline{i(\overline{B_1}^X)}^{w^*}$$

□