Functional Analysis II, Math 7321
Lecture Notes from February 16, 2017

taken by Nikolaos Karantzas

2.D Duality and Closed range

We begin with a corollary of the Hahn-Banach theorem.

2.56 Corollary. *Suppose B is a convex, balanced, closed set in a locally convex space X, and let $x_0 \in X$, but $x_0 \in B$. Then there exists $\Lambda \in X'$ such that $|\Lambda x| \le 1$ for all $x \in B$, but $\Lambda x_0 > 1$.*

Proof. Since B is closed and convex, we apply Theorem 4.2.1 [15 November 2016], with $A = \{x_0\}$, to obtain $\Lambda_1 \in X'$ such that $\Lambda_1 x_0 = re^{i\theta}$ lies outside the closure K of $\Lambda_1(B)$. Since B is balanced, so is K. Hence there exists s, with $0 < s < r$, such that $|z| \le x$ for all $z \in K$. The functional $\Lambda = s^{-1}e^{-i\theta}\Lambda_1$ has the desired properties. □

Next, we recall that if X and Y are Banach spaces and if $T \in B(X, Y)$, then by "generalized" rank-nullity,

$$\overline{ran(T)} = ker(T')^{\perp}.$$

Thus, $T(X)$ is dense in Y if and only if T' is injective. Our goal is to lay the groundwork for establishing a condition for surjectivity of T in terms of T'. We prepare this with the following lemma.

2.57 Lemma. *Let X and Y be Banach spaces and let $T \in B(X, Y)$ and $r > 0$. Then, we have the following:*

(a) If $B_r^Y \subset \overline{T(B_1^X)}$, then $B_r^Y \subset T(B_1^X)$.

(b) If $\|T'f\| \ge r\|f\|$ for all $f \in Y$, then $B_r^Y \subset T(B_1^X)$.

Proof. (a). We assume, without loss of generality, that $r = 1$, since otherwise, $\overline{T(B_1^X)}$ is balanced and so we can scale accordingly. Then if $B_1^Y \subset \overline{T(B_1^X)}$, the same inclusion holds for the closure, i.e,

$$\overline{B_1^Y} \subset \overline{T(B_1^X)}.$$

Now for a given $y \in B_1^Y$ and any $\epsilon > 0$, the above inclusion implies that there exists $x \in B_1^X$ such that $\|x\| \le \|y\|$ and $\|y - Tx\| < \epsilon$. For any such $y \in B_1^Y$, let $y_1 = y$ and take a sequence $\{\epsilon_n\}_{n=1}^{\infty}$, with $\epsilon_n > 0$, such that

$$\sum_{n=1}^{\infty} \epsilon_n < 1 - \|y_1\|.$$

Next, for any $n \in \mathbb{N}$, given $y_n \in B_1^Y$, there exists $x_n \in B_1^X$ with $\|x_n\| \le \|y_n\|$ and

$$\|y_n - Tx_n\| < \epsilon_n.$$

So, setting $y_{n+1} = y_n - Tx_n$ defines a pair of sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ with

$$\|x_{n+1}\| \le \|y_{n+1}\| = \|y_n - Tx_n\| < \epsilon_n.$$

Thus,

$$\sum_{n=1}^{\infty} \|x_n\| \le \|x_1\| + \sum_{n=1}^{\infty} \epsilon_n \le \|y_1\| + \sum_{n=1}^{\infty} \epsilon_n < \|y_1\| + 1 - \|y_1\| = 1.$$

Now, since $\sum_{n=1}^{\infty} \|x_n\|$ is convergent, we have that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k > m > N$, $\sum_{n=m+1}^{k} \|x_n\| < \epsilon$. Therefore, the sequence of partial sums $S_k = \sum_{n=1}^{k} x_n$ satisfies

$$\|S_k - S_m\| = \left\| \sum_{n=m+1}^{k} x_n \right\| \le \sum_{n=m+1}^{n} \|x_n\| < \epsilon.$$

Thus, S_k is Cauchy and since X is Banach, there exists $x \in X$, such that $x = \sum_{n=1}^{\infty} x_n$. Moreover, $x \in B_1^X$, and

$$Tx = \lim_{N\to\infty} \sum_{n=1}^{N} Tx_n = \lim_{N\to\infty} \sum_{n=1}^{N} (y_n - y_{n+1}) = y_1.$$

Besides the fact that the finite sum $\sum_{n=1}^{N}(y_n - y_{n+1})$ is telescoping, the above equality holds because $x_n \to 0$ implies $Tx_n \to 0$ and so $\epsilon_n \to 0$ implies $y_n \to 0$. So we have shown that for every $y \in B_1^Y$, there is an $x \in B_1^X$, such that $Tx = y$, which means $B_1^Y \subset T(B_1^X)$.

(b). We only need to show that $B_r^Y \subset \overline{T(B_1^X)}$, because then we can apply (a) to get the claimed inclusion. We prove the equivalent inclusion for the complements. To this end, we pick $y_0 \notin \overline{T(B_1^X)}$. Then, by convexity, closedness, and balandedness, Corollary 3.4.24 implies we have strict separation by some $f \in Y'$, with $|\langle f, y\rangle| \le 1$ for every $y \in \overline{B_1^X}$, but $|\langle f, y_0\rangle| > 1$. If $x \in B_1^X$, it follows that

$$|\langle x, T'f\rangle| = |\langle Tx, f\rangle| \le 1.$$

Hence, $\|T'f\| \le 1$, and so our assumption implies

$$r < r|\langle f, y_0\rangle| \le r\|f\|\|y_0\| \le \|T'f\|\|y_0\| \le \|y_0\|,$$

which means $y_0 \notin B_r^Y$. By taking the complements, we get $B_r^Y \subset \overline{T(B_1^X)}$, so now applying (a) finishes the proof. □

2.58 Theorem. *Let X, Y be Banach spaces and let $T \in B(X,Y)$. Then the following assertions are equivalent:*

1. *$ran(T)$ is closed in Y.*

2. *$ran(T')$ is weak-$*$ closed in X'.*

3. $ran(T')$ *is closed in* X'*.*

Proof. We prove how we get (3) assuming (2). From $i(X) \subset X''$, the weak-$*$ topology on X' is the initial topology induced by $i(X)$, as an initial topology which is coarser than the weak topology on X' induced by X'', which in turn is coarser than the norm topology on X'. This means that if a set is weak-$*$ closed in X', then it is closed in X'. $\square$

References

[1] W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.