Last Time

- Duality and Closed range
- Towards a characterization of surjectivity of $T$ in terms of $T'$

Review: Injectivity vs. Invertibility

Given $X$, $Y$ be normed spaces,

(i) Let $T \in B(X,Y)$, if $T$ is injective and surjective (i.e. bijective) with the inverse in $B(Y,X)$, then $T$ is invertible.

(ii) If $T \in B(X,Y)$ is invertible, from the definition, we know that there exists $S \in B(Y,X)$ such that $ST = I_X$, $TS = I_Y$. Then $T$ is one-to-one and $\exists \delta > 0$, $B_{\delta}^Y \subset T(B_1^X)$. We have a consequence that $\exists \delta > 0$: \[
\inf_{\|x\|=1} \|Tx\| \geq \delta
\]
So $T$ is injective. Note that the norm bound is a consequence of invertibility, $T$ does not need to be surjective, it is a weaker property.

We recall $\text{ran}(T)$ is dense in $Y$ if and only if $\text{ran}(T) \perp = \{0\}$; in that case, $\text{ker}(T') = \text{ran}(T) \perp = \{0\}$. We have $\text{ran}(T) = (\text{ker}(T')) \perp = Y$.

So $\text{ran}(T)$ is dense in $Y$ if and only if $T'$ is injective.

2.59 Problem. Can we find the condition for $\text{ran}(T) = Y$ in terms of $T'$?

Suppose $X$ and $Y$ are Banach spaces, and $T \in B(X,Y)$, then $\text{ran}(T) = Y$ if and only if $T'$ is injective and $\text{ran}(T')$ is norm-closed.

Warm-up:

Let $X$ and $Y$ are Banach spaces,

(i) If $T \in B(X,Y)$ is invertible, then $T': Y' \to X'$ is invertible.

Proof. If $I_X$ and $I_Y$ are the identity mappings on $X$ and $Y$, respectively, then their duals mappings are the same as the identity mappings $I_{X'}$ and $I_{Y'}$ on $X'$ and $Y'$, respectively. Thus $T^{-1} \circ T = I_X$. 
and
\[ T \circ T^{-1} = I_Y \]
we get that
\[ T' \circ (T^{-1})' = (T^{-1} \circ T)' = I_{X'}, \]
and
\[ (T^{-1})' \circ T' = (T \circ T^{-1})' = I_{Y'} \]
So \((T')^{-1} \in B(X', Y')\) and \((T')^{-1} = (T^{-1})'\). Hence, \(T'\) is invertible.

\(\square\)

(ii) If \(T'\) is invertible, \(T\) is invertible.

**Proof.** From \(T'\) is invertible, we have \(T'' : X'' \to Y''\) is invertible.

Consider the natural (Canonical) map \(i : X \to X'', i(x)(f) = f(x)\) for \(x \in X, f \in X'\).

Clearly \(\|i(x)\| \leq \|f\|\) and, by the Hahn-Banach theorem, equality holds. Frequently, \(X\) is identified with \(i(X)\), then \(X\) is regarded as a subspace of \(X''\).

This mapping is isometric and therefore bounded:
\[
\|i(x)\| = \sup_{f \in S_{X'}} |i(x)(f)| = \sup_{f \in S_{X'}} |f(x)| = \|x\|
\]
for every vector \(x \in X\). This implies that \(i\) is injective: If \(i(x) = 0\), then \(\|x\| = \|i(x)\| = 0\), and therefore \(x = 0\).

Notice that \(X\) is isometrically isomorphic to the image \(i(X)\) of \(X\) under the natural (canonical) embedding: \(X \cong i(X)\).

If \(X\) is reflexive, then \(X\) is thus isometrically isomorphic to \(X''\) via the natural embedding. This means that any linear functional \(F \in X''\) has the form \(F = i(x)\) for some vector \(x \in X\), i.e., \(F(f) = f(x)\) for every linear functional \(f \in X'\).

Thus, if \(X\) and \(Y\) are reflexive, then it is easy to see that \(T''\) corresponds exactly to \(T\) under the natural isomorphisms between \(X\) and \(X''\) and \(Y\) and \(Y''\), and hence that \(T\) is invertible.

Otherwise, \(T\) corresponds to the restriction of \(T''\) to the image of the natural embedding of \(X\) into \(X''\), which takes values in the image of the natural embedding of \(Y\) in \(Y''\). This implies that
\[
\|T(x)\|_Y \geq \delta \|x\|_X
\]
for some \(\delta > 0\), and every \(x \in X\), because of the analogous condition for \(T''\) that follows from invertibility.

From \(\text{ran}(T) = (\ker(T'))^\perp\), we know that \(T(X)\) is dense in \(Y\) if and only if \(T'\) is injective. \(X\) is complete, we have \(T(X)\) is complete as well. So \(T(X)\) is a closed linear subspace of \(Y\). If \(T'\) is invertible, then \(\ker(T') = \{0\}\), so that \(T(X)\) is dense in \(Y\). Thus we get that \(T(X) = Y\) under these conditions, because \(T(X)\) is both dense and closed in \(Y\). This shows that \(T : X \to Y\) is invertible when \(T' : Y' \to X'\) is invertible, as desired.

\(\square\)

**2.60 Proposition.** Given \(T \in B(X, Y)\), and \(T'\) is invertible, then \(T\) satisfies
\[
\inf_{\|x\|=1} \|T(x)\| > 0.
\]
Proof. We know from $T'$ invertible, then $T'' : X'' \to Y''$ is invertible, so $T''$ satisfies

$$\inf_{\|x''\| = 1} \|T''x''\| > 0.$$ 

(From above warm-up (ii), we know that for $T \in B(X,Y)$, $T'$ is invertible, then $T$ is invertible.) By $i(X) \subset X''$, $T''|_{i(X)} \cong T$, so $T$ satisfies the norm bound.

We had stated:

2.61 Theorem. If $X$ and $Y$ are Banach spaces, let $T \in B(X,Y)$, then the following assertions are equivalent:

(1) $\text{ran}(T)$ is closed in $Y$.
(2) $\text{ran}(T')$ is weak-* closed in $X'$.
(3) $\text{ran}(T')$ is closed in $X'$.

Proof. (2) $\Rightarrow$ (3) was proved last time.

We prove (1) $\Rightarrow$ (2). Assume (1) holds, then we know

$$\ker(T) = \{ f \in X' : f(x) = 0 \text{ for each } x \in \ker(T) \}$$

$$= \bigcap_{x \in \ker(T)} \{ f \in X' : f(x) = 0 \} \quad (\text{we have } i(x)(f) = 0)$$

$$= \bigcap_{x \in \ker(T)} \ker i(x) \quad (\text{weak-* closed})$$

$$= \bigcap_{x \in \ker(T)} \ker i(x)^{w^*}$$

$$= \overline{\ker(T)}^{w^*}$$

By generalized rank-nullity,

$$\ker(T)^\perp = \overline{\text{ran}(T')} = \overline{\text{ran}(T')^{w^*}}.$$

It is left to show $\ker(T)^\perp \subset \text{ran}(T')$.

Let $f \in \ker(T)^\perp$. Define $g$ on $\text{ran}(T)$ by $g(Tx) = \langle f, x \rangle$. This is well defined because if $Tx = T x'$, then $x - x' \in \ker(T)$, so $\langle f, x - x' \rangle = 0$ and $\langle f, x \rangle = \langle f, x' \rangle$.

Using the open mapping theorem, $T : X \to \text{ran}(T)$ is onto a complete space since $\text{ran}(T)$ is closed, so $T$ is open, hence there is $\delta > 0$, such that $T(B_1^X) \supset B_{\delta}^{\text{ran}(T)}$ and for $g$ defined above

$$|g(y)| = |g(Tx)| = |\langle f, x \rangle| \leq \|f\|\|x\| \leq \frac{1}{\delta}\|f\|\|y\|.$$ 

Hence $g$ is continuous on the range of $T$ and extends by Hahn Banach to $G$ on $Y'$. Thus,

$$\langle G, Tx \rangle = g(Tx) = \langle f, x \rangle$$

for $x \in X$.

Thus, $T'G = f$. Since $f$ was arbitrary in $\ker(T)^\perp$, we see $\ker(T)^\perp \subset \text{ran}(T')$. By continuing
inclusions, \( \ker(T)^\perp = \text{ran}(T') \). Thus, \( \text{ran}(T') \) is weak-* closed. Finally, we show (3) \( \Rightarrow \) (1). Let \( Z = \text{ran}(T) \). Let \( S \in B(X, Z) \), \( Sx = Tx \), then \( \text{ran}(S) = Z \), so \( S' : Z' \to X' \) is injective by \( \ker(S')^\perp = \text{ran}(S) \).

For \( f \in Z' \), we get by Hahn Banach finding \( F \) in \( Y' \) such that for each \( x \in X \),

\[
\langle T'F, x \rangle = \langle F, Tx \rangle = \langle f, Sx \rangle = \langle S'f, x \rangle
\]

so \( S'f = T'F \), \( \text{ran}(S') = \text{ran}(T') \).

By assumption on \( \text{ran}(T') \) being closed, so is \( \text{ran}(S') \) and hence \( \text{ran}(S') \) is complete, so by the open mapping theorem, for \( S' : Z' \to \text{ran}(S') \) there is \( \delta > 0 \) such that for each \( h \in Z' \), \( \|S'h\| \geq \delta \|h\| \). Hence, by our warm-up exercise, \( S : X \to Z \) is open as well, so \( S(X) = Z \), but \( \text{ran}(T) = \text{ran}(S) \), so \( \text{ran}(T) = Z \) is closed in \( Y \).

We are ready to characterize surjectivity of \( T \).

2.62 Theorem. Let \( X, Y \) be Banach spaces, \( T \in B(X, Y) \), then \( \text{ran}(T) = Y \) if and only if there is \( \delta > 0 \) such that \( \|T'f\| \geq \delta \|f\| \) for all \( f \in Y' \).

Proof. We know that \( T \) is surjective if and only if \( \text{ran}(T) \) is dense and (norm) closed in \( Y \). By the closed range characterization, we have that \( \text{ran}(T) \) is dense in \( Y \) if and only if \( T' \) is injective. So it is equivalent to \( T \) is surjective if and only if \( T' \) being injective and \( \text{ran}(T) \) (norm) closed in \( Y \). The closedness of \( \text{ran}(T) \), in turn, is equivalent to \( T' \) being norm bounded below. Thus, \( \text{ran}(T) = Y \) if and only if \( T' \) is injective and \( \text{ran}(T') \) is norm-closed.

(a) We know \( T' \) is injective. By the open mapping theorem there is \( \delta > 0 \) such that

\[
\{ y \in Y \mid \|y\| \leq \delta \} \subset \{ T(x) \mid \|x\| \leq 1 \}.
\]

Then for a functional \( f \),

\[
\|T'f\| = \sup\{ |(T'f)(x)| \mid \|x\| \leq 1 \}
\]

\[
= \sup\{ |f(Tx)| \mid \|x\| \leq 1 \}
\]

\[
\geq \sup\{ |f(y)| \mid \|y\| \leq \delta \}
\]

\[
= \delta \|f\|.
\]

We claim that given this inequality, \( \text{ran}(T') \) is closed.

(b) By Theorem 1.2.2, \( \text{ran}(T) \) is closed. And it is dense, so \( \text{ran}(T) = Y \).