Functional Analysis II, Math 7321
Lecture Notes from February 21, 2017

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Last Time

- Duality and Closed range
- Towards a characterization of surjectivity of T in terms of T'

Review: Injectivity $vs.$ Invertibility

Given X, Y be normed spaces,

(i) Let $T \in B(X,Y)$, if T is injective and surjective (i.e. bijective) with the inverse in $B(Y,X)$, then T is invertible.

(ii) If $T \in B(X,Y)$ is invertible, from the definition, we know that there exists $S \in B(Y,X)$ such that $ST = I_X$, $TS = I_Y$. Then T is one-to-one and $\exists \delta > 0$, $B_\delta^Y \subset T(B_1^X)$. We have a consequence that $\exists \delta > 0$:

$$\inf_{\|x\|=1} \|Tx\| \geqslant \delta$$

So T is injective. Note that the norm bound is a consequence of invertibility, T does not need to be surjective, it is a weaker property.

We recall $ran(T)$ is dense in Y if and only if $ran(T)^\perp = \{0\}$; in that case, $ker(T') = ran(T)^\perp = \{0\}$. We have

$$\overline{ran(T)} = (ker(T'))^\perp = Y.$$

So $ran(T)$ is dense in Y if and only if T' is injective.

2.59 Problem. Can we find the condition for $ran(T) = Y$ in terms of T'?

Suppose X and Y are Banach spaces, and $T \in B(X,Y)$, then $ran(T) = Y$ if and only if T' is injective and $ran(T')$ is norm-closed.

Warm-up:

Let X and Y are Banach spaces,

(i) If $T \in B(X,Y)$ is invertible, then $T' : Y' \to X'$ is invertible.

Proof. If I_X and I_Y are the identity mappings on X and Y, respectively, then their duals mappings are the same as the identity mappings $I_{X'}$ and $I_{Y'}$ on X' and Y', respectively. Thus

$$T^{-1} \circ T = I_X,$$

and

$$T \circ T^{-1} = I_Y$$

we get that

$$T' \circ (T^{-1})' = (T^{-1} \circ T)' = I_{X'},$$

and

$$(T^{-1})' \circ T' = (T \circ T^{-1})' = I_{Y'}$$

So $(T')^{-1} \in B(X', Y')$ and $(T')^{-1} = (T^{-1})'$. Hence, T' is invertible.

□

(ii) If T' is invertible, T is invertible.

Proof. From T' is invertible, we have $T'' : X'' \to Y''$ is invertible.
Consider the natural (Canonical) map $i : X \to X''$, $i(x)(f) = f(x)$ for $x \in X$, $f \in X'$.
Clearly $\|i(x)\| \leqslant \|f\|$ and, by the Hahn-Banach theorem, equality holds. Frequently, X is identified with $i(X)$, then X is regarded as a subspace of X''.
This mapping is isometric and therefore bounded:

$$\|i(x)\| = \sup_{f \in S_{X'}} |i(x)(f)| = \sup_{f \in S_{X'}} |f(x)| = \|x\|$$

for every vector $x \in X$. This implies that i is injective: If $i(x) = 0$, then $\|x\| = \|i(x)\| = 0$, and therefore $x = 0$.
Notice that X is isometrically isomorphic to the image $i(X)$ of X under the natural (canonical) embedding: $X \cong i(X)$.
If X is reflexive, then X is thus isometrically isomorphic to X'' via the natural embedding. This means that any linear functional $F \in X''$ has the form $F = i(x)$ for some vector $x \in X$, i.e., $F(f) = f(x)$ for every linear functional $f \in X'$.
Thus, if X and Y are reflexive, then it is easy to see that T'' corresponds exactly to T under the natural isomorphisms between X and X'' and Y and Y'', and hence that T is invertible. Otherwise, T corresponds to the restriction of T'' to the image of the natural embedding of X into X'', which takes values in the image of the natural embedding of Y in Y''. This implies that

$$\|T(x)\|_Y \geqslant \delta \|x\|_X$$

for some $\delta > 0$, and every $x \in X$, because of the analogous condition for T'' that follows from invertibility.
From $\overline{ran(T)} = (ker(T'))^{\perp}$, we know that $T(X)$ is dense in Y if and only if T' is injective. X is complete, we have $T(X)$ is complete as well. So $T(X)$ is a closed linear subspace of Y. If T' is invertible, then $ker(T') = \{0\}$, so that $T(X)$ is dense in Y. Thus we get that $T(X) = Y$ under these conditions, because $T(X)$ is both dense and closed in Y. This shows that $T : X \to Y$ is invertible when $T' : Y' \to X'$ is invertible, as desired.

□

2.60 Proposition. *Given $T \in B(X, Y)$, and T' is invertible, then T satisfies*

$$\inf_{\|x\|=1} \|Tx\| > 0.$$

Proof. We know from T' invertible, then $T'' : X'' \to Y''$ is invertible, so T'' satisfies

$$\inf_{\|x''\|=1} \|T''x''\| > 0.$$

(From above warm-up (ii), we know that for $T \in B(X,Y)$, T' is invertible, then T is invertible.) By $i(X) \subset X''$, $T''|_{i(X)} \cong T$, so T satisfies the norm bound.

□

We had stated:

2.61 Theorem. *If X and Y are Banach spaces, let $T \in B(X,Y)$, then the following assertions are equivalent:*
(1) $ran(T)$ is closed in Y.
(2) $ran(T')$ is weak-$*$ closed in X'.
(3) $ran(T')$ is closed in X'.

Proof. (2) ⇒ (3) was proved last time.
We prove (1) ⇒ (2). Assume (1) holds, then we know

$$\begin{aligned}
ker(T)^{\perp} &= \{f \in X' : f(x) = 0 \text{ for each } x \in ker(T)\} \\
&= \bigcap_{x \in ker(T)} \{f \in X' : f(x) = 0\} \qquad \text{(we have } i(x)(f) = 0\text{)} \\
&= \bigcap_{x \in ker(T)} ker\ i(x) \qquad \text{(weak-}*\text{ closed)} \\
&= \overline{\bigcap_{x \in ker(T)} ker\ i(x)}^{w^*} \\
&= \overline{ker(T)^{\perp}}^{w^*}
\end{aligned}$$

By generalized rank-nullity,

$$ker(T)^{\perp} = \overline{ran(T')} = \overline{ran(T')}^{w^*}.$$

It is left to show $ker(T)^{\perp} \subset ran(T')$.
Let $f \in ker(T)^{\perp}$. Define g on $ran(T)$ by $g(Tx) = \langle f, x \rangle$. This is well defined because if $Tx = Tx'$, then $x - x' \in ker(T)$, so $\langle f, x - x' \rangle = 0$ and $\langle f, x \rangle = \langle f, x' \rangle$.
Using the open mapping theorem, $T : X \to ran(T)$ is onto a complete space since $ran(T)$ is closed, so T is open, hence there is $\delta > 0$, such that $T(B_1^X) \supset B_\delta^{ran(T)}$ and for g defined above

$$|g(y)| = |g(Tx)| = |\langle f, x \rangle| \leqslant \|f\|\|x\| \leqslant \frac{1}{\delta}\|f\|\|y\|.$$

Hence g is continuous on the range of T and extends by Hahn Banach to G on Y'. Thus,

$$\langle G, Tx \rangle = g(Tx) = \langle f, x \rangle$$

for $x \in X$.
Thus, $T'G = f$. Since f was arbitrary in $ker(T)^{\perp}$, we see $ker(T)^{\perp} \subset ran(T')$. By continuing

inclusions, $ker(T)^\perp = \underline{ran(T')}$. Thus, $ran(T')$ is weak-$*$ closed.
Finally, we show (3) $\Rightarrow$ (1). Let $Z = \overline{ran(T)}$. Let $S \in B(X, Z)$, $Sx = Tx$, then $\overline{ran(S)} = Z$, so $S' : Z' \to X'$ is injective by $ker(S')^\perp = \overline{ran(S)}$.
For $f \in Z'$, we get by Hahn Banach F in Y' such that for each $x \in X$,

$$\langle T'F, x\rangle = \langle F, Tx\rangle = \langle f, Sx\rangle = \langle S'f, x\rangle$$

so $S'f = T'F$, $ran(S') = ran(T')$.
By assumption on $ran(T')$ being closed, so is $ran(S')$ and hence $ran(S')$ is complete, so by the open mapping theorem, for $S' : Z' \to ran(S')$ there is $\delta > 0$ such that for each $h \in Z'$, $\|S'h\| \geqslant \delta\|h\|$. Hence, by our warm-up exercise, $S : X \to Z$ is open as well, so $S(X) = Z$, but $ran(T) = ran(S)$, so $ran(T) = Z$ is closed in Y.

□

We are ready to characterize surjectivity of T.

2.62 Theorem. *Let X, Y be Banach spaces, $T \in B(X, Y)$, then $ran(T) = Y$ if and only if there is $\delta > 0$ such that $\|T'f\| \geqslant \delta\|f\|$ for all $f \in Y'$.*

Proof. We know that T is surjective if and only if $ran(T)$ is dense and (norm) closed in Y.
By the closed range characterization, we have that $ran(T)$ is dense in Y if and only if T' is injective.
So it is equivalent to T is surjective if and only if T' being injective and $ran(T)$ (norm) closed in Y. The closedness of $ran(T)$, in turn, is equivalent to T' being norm bounded below.
Thus, $ran(T) = Y$ if and only if T' is injective and $ran(T')$ is norm-closed.
(a) We know T' is injective. By the open mapping theorem there is $\delta > 0$ such that

$$\{y \in Y \mid \|y\| \leqslant \delta\} \subset \{T(x) \mid \|x\| \leqslant 1\}.$$

Then for a functional f,

$$\begin{aligned}\|T'f\| &= \sup\{|(T'f)(x)| \mid \|x\| \leqslant 1\}\\ &= \sup\{|f(Tx)| \mid \|x\| \leqslant 1\}\\ &\geqslant \sup\{|f(y)| \mid \|y\| \leqslant \delta\}\\ &= \delta\|f\|.\end{aligned}$$

We claim that given this inequality, $ran(T')$ is closed.
(b) By Theorem 1.2.2, $ran(T)$ is closed. And it is dense, so $ran(T) = Y$. □