Functional Analysis, Math 7321
Lecture Notes from February 28, 2017

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Warm up: Non-commuting fractions.

3.9 Theorem. *(Resolvent identify) Consider $R_c(T) = (cI - T)^{-1}$ and assuming $R_c(T+S)$ exits, then*

$$R_c(T+S) = R_c(T) + R_c(T)SR_c(T+S).$$

Proof. To show this identity, we first consider

$$cI - T = cI - T - S + S.$$

Multiplying both side by $R_c(T)$ we get,

$$R_c(T)(cI - T) = R_c(T)(cI - T - S + S).$$

Since $R_c(T) = (cI - T)^{-1} \Rightarrow R_c(T)(cI - T) = I$. Thus,

$$I = R_c(T)(cI - T - S + S) = R_c(T)(cI - T - S) + R_c(T)S$$

Again multiplying both side by $R_c(T+S)$ we get,

$$\begin{aligned} R_c(T+S) &= R_c(T+S)(R_c(T)(cI - (T+S)) + R_c(T)S) \\ &= R_c(T) + R_c(T+S)SR_c(T) \quad (\text{since } \ R_c(T+S) = (cI - (T+S))^{-1}) \end{aligned}$$

Hence the identity is derived. □

3.10 Theorem. *If $r(T) < 1$ then $I - T$ is bijective operator from X to X with bounded inverse*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k,$$

where the (Neumann series) series converges with respect to the norm of $B(X)$.

Proof. We have already shown that $\|T\| < 1$.Clearly $\sum_{k=0}^{\infty} \|T^k\| < \infty$ (By Root test). Since $B(X)$ is complete, then by the completeness the series $S = \sum_{k=0}^{\infty} T^k$ is convergent. Now,

$$(I - T)S = (I - T)\sum_{k=0}^{\infty} T^k = \sum_{k=0}^{\infty}(T^k - T^{k+1}) = (I - T) + (T - T^2) + ... = I$$

Similarly, we can show that $S(I - T) = I$. Which means the operator $I - T$ is a bijective operator on X and that its inverse is given by the Neumann series S defined above. Again, since for every $x \in X$ we have $\|Sx\| \leq (\sum_{k=0}^{\infty} \|T^k\|)\|x\|$ and $(I - T)^{-1}$ is bounded on X with norm less or equal to $\sum_{k=0}^{\infty} \|T^k\|$ □

Now, from above theorem we can say that if $\|R_c(T)\|\|S\| < 1$ then the series obtained from iterating Resolvent identity converges with respect to the operator norm $\|.\|$ on $B(X)$ and gives

$$\begin{aligned} R_c(T+S) &= R_c(T) + R_c(T+S)SR_c(T) \\ (1 - R_c(T)S)R_c(T+S) &= R_c(T) \\ R_c(T+S) &= R_c(T)(1 - SR_c(T))^{-1} \\ &= R_c(T) + \sum_{j=1}^{\infty} R_c(T)(SR_c(T))^j \\ &= \sum_{j=0}^{\infty} S^j (R_c(T))^{j+1} \end{aligned}$$

In the special case $S = (c-w)I$ with

$$|c-w| < \|R_c(T)\|^{-1}$$

This results in

$$\begin{aligned} R_w(T) &= R_c(T + (c-w)I) \\ &= \sum_{j=0}^{\infty} (c-w)^j (R_c(T))^{j+1}. \end{aligned}$$

Now we have the spectral radius $r = \lim_{n\to\infty} \sup \|T^n\|^{\frac{1}{n}}$ and we have from the lecture note on February 2, 2017 the resolvent set of T is $\rho(T) = \{c \in \mathbb{C} : T - CI \text{ is invertible}\}$ and $\rho(T) = \mathbb{C} \backslash \sigma(T)$ we can see from the sketch below

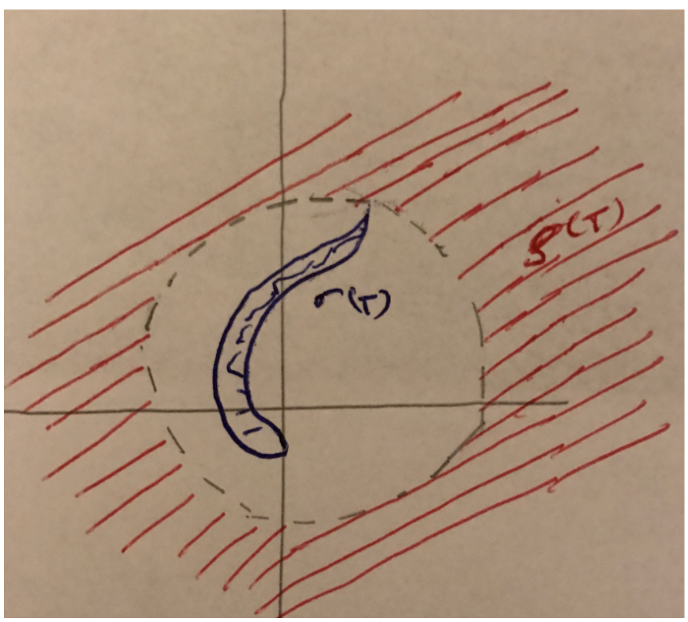

Figure 1: Sketch

3.11 Theorem. *For $T \in B(x)$,*

$$max\{|z| : z \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$$

Proof. Let $r = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ then we show there is $z \in \sigma(T), |z| = r$.

if $r = 0$ then from Gelfand theorem (from the lecture note on February 23, 2017) we have

$$\phi \neq \sigma(T) \subset \overline{B}_0(0) \Rightarrow \sigma(T) = \{0\}.$$

Next, consider $r > 0$. Assume $\sigma(T) \cap \{z : |z| = r\} = \phi$. Then,

$$max\{|z| : z \in \sigma(T)\} < r.$$

Take $R > 0$ such that

$$r(T) = max\{|z| : z \in \sigma(T)\} < R < r$$

Then,

$$\sigma(T) \subset \overline{B}_{r(T)}(0)$$

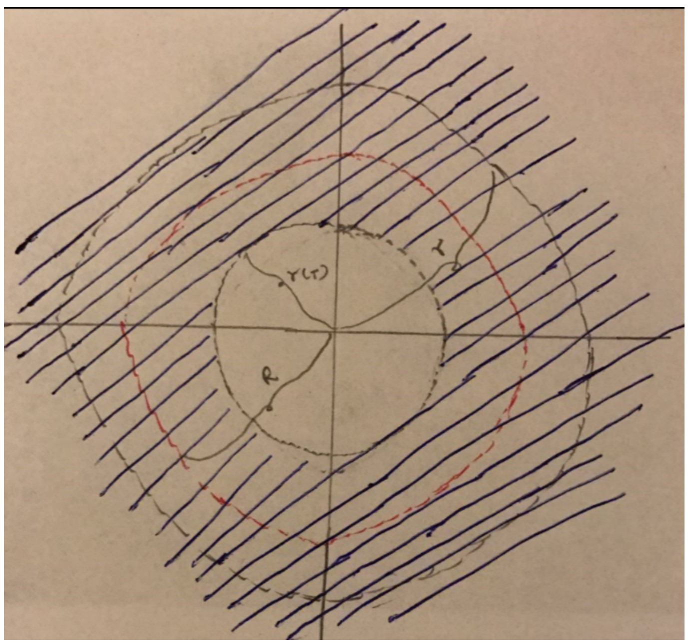

Figure 2: Sketch

By the series computation from the proof of Warm of theorem above (Theorem 6.1.2), for $f \in B(x)'$ then $g(z) = f((T - zI)^{-1})$ defines a holomorphic function g on $\rho(T) \supset \{z : |z| > r(T)\}$ with

$$g(z) = -\sum_{n=0}^{\infty} f(T^n) z^{-(n+1)}$$

The domain of analyticity includes $\{z \in \ : |z| = R\}$ so

$$\sup_{n\geq 0} \frac{|f(T^n)|}{R^{n+1}} < \infty. \text{ (since the series is convergent)}$$

This is true for any $f \in B(x)'$ with $\|f\| \leq 1$. So from uniform boundedness (from the lecture note January 31, 2017).

$$c = \sup_{n \geq 0} \|\frac{T^n}{R^{n+1}}\| < \infty \text{ or } \|T^n\| \leq cR^{n+1}$$

And thus,

$$\|T^n\|^{\frac{1}{n}} \leq c^{\frac{1}{n}} R^{1+\frac{1}{n}}$$

So $\lim_{n\to\infty} \sup \|T^n\|^{\frac{1}{n}} = r \leq R$, which contradicts with our assumption $R < r$.

Next, we show

$$r = \inf\{\|T^n\|^{\frac{1}{n}}\}.$$

Let $n, m \in \mathbb{N}$ then $n = qm + k,\ \ k \in \{0, 1, 2, ..., m-1\}$ and by the fundamental norm inequality for operator norm we have,

$$\|T^n\| \leq \|T^m\|^{\frac{q}{n}} \ \ \|T\|^{\frac{k}{n}}$$

so

$$\|T^n\|^{\frac{1}{n}} \leq \|T^m\|^q \ \ \|T\|^k$$

Fixing m and letting $n \to \infty$, by $n = qm + k \to 1 = \frac{qm}{n} + \frac{k}{n}$ we get $\frac{k}{n} \overset{n\to\infty}{\to} 0$, $\frac{q}{n} \to \frac{1}{m}$ so,

$$r = \lim_{n\to\infty} \sup \|T^n\|^{\frac{1}{n}} \leq \|T^m\|^{\frac{1}{m}}$$

Then taking the infimum over $m \in \mathbb{N}$, we get

$$\begin{aligned} r &\leq \inf\{\|T^m\|^{\frac{1}{m}}\}_{m=1}^{\infty} \\ &\leq \lim_{m\to\infty} \inf \|T^m\|^{\frac{1}{m}} \\ &\leq \lim_{m\to\infty} \sup \|T^m\|^{\frac{1}{m}} = r \end{aligned}$$

Hence the limit exits and equality holds throughout. □

3.12 Example. Let $T : l^1 \to l^1$ be defined by $T(x_1, x_2, ...,) = (x_2, x_3, ...)$ then

$$\|T(x_1, x_2, ...,)\|_1 = \|(x_1, x_2, ...,)\|_1 = \sum_{j=2}^{\infty} |x_j| \leq \sum_{j=2}^{\infty} |x_j| = \|x\|_1$$

Hence T is contraction.

Also, setting $x_1 = 0$ shows $\|T\| = 1$, then by above theorem, $\sigma(T) \subset \overline{B}_{r(T)}(0)$, and

$$r = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \|T\| = 1$$

We show $r(T) = 1$, we see that if $|z| < 1$, then

$$T(1, z, z^2, ...) = (z, z^2, z^3, ...) = z(1, z, z^2, ...)$$

So $z \in \sigma(T) \cong ker(T - zI)$ and hence

$$B_1(0) \subset \sigma(T) \subset \overline{B}_1(0).$$

We conclude by closeness of $\sigma(T)$, $\sigma(T) = \overline{B}_1(0)$.

3.13 Example. It is possible for T to be injective $\sigma(T) = \{0\}$. Let $C([0,1])$ is equipped with $\|.\|_\infty$, and let T be given by

$$T : C([0,1]) \to C([0,1]) \text{ be } Tf(x) = \int_0^x f(t)dt$$

Then,

$$\begin{aligned}\|Tf\|_\infty &\leq \sup_{x\in[0,1]} \int_0^x |f(t)|dt \\ &\leq \int_0^1 \|f\|_\infty dt \\ &= \|f\|_\infty\end{aligned}$$

So T is contraction.

For $f = 1$, $\|Tf\|_\infty = 1$, $\|T\| = 1$.

Next, we see

$$|T^n f(x)| \leq \|f\|_\infty \frac{x^n}{n!}$$

Let $n = 1$, then $|Tf(x)| \leq \|f\|_\infty x$. Assuming, inequality holds for $n \in \mathbb{N}$, then

$$\begin{aligned}|(T^{n+1}f)x| &= |\int_0^x T^n f(t)dt| \\ &\leq \int_0^x |T^n f(t)|dt \\ &\leq \int_0^x |T^n f(t)|dt \text{ (by induction assumption)} \\ &= \|f\|_\infty \frac{x^{n+1}}{(n+1)!}\end{aligned}$$

Consequentially,

$$\|T^n f\|_\infty = \|f\|_\infty \frac{1}{n!}$$

And for $f = 1$, we get equality. Thus,

$$\|T^n\|^{\frac{1}{n}} = (\frac{1}{n!})^{\frac{1}{n}} \overset{n\to\infty}{\to} 0.$$

So, $r(T) = 0$ Hence, T is injective, but the spectrum is the same as that of the zero map this means $\sigma(T) = 0$.