Warm up: Non-commuting fractions.

3.9 Theorem. (Resolvent identity) Consider \( R_c(T) = (cI - T)^{-1} \) and assuming \( R_c(T + S) \) exits, then
\[
R_c(T + S) = R_c(T) + R_c(T)SR_c(T + S).
\]

Proof. To show this identity, we first consider
\[
cI - T = cI - (T + S) + S.
\]
Multiplying both side by \( R_c(T) \) we get,
\[
R_c(T)(cI - T) = R_c(T)(cI - T - S) + R_c(T)S.
\]
Since \( R_c(T) = (cI - T)^{-1} \Rightarrow R_c(T)(cI - T) = I. \) Thus,
\[
I = R_c(T)(cI - T - S + S) = R_c(T)(cI - T - S) + R_c(T)S.
\]
Again multiplying both side by \( R_c(T + S) \) we get,
\[
R_c(T + S) = R_c(T + S)(R_c(T)(cI - (T + S)) + R_c(T)S) = R_c(T) + R_c(T + S)SR_c(T) \quad \text{(since } R_c(T + S) = (cI - (T + S))^{-1})
\]
Hence the identity is derived.

3.10 Theorem. If \( r(T) < 1 \) then \( I - T \) is bijective operator from \( X \) to \( X \) with bounded inverse
\[
(I - T)^{-1} = \sum_{k=0}^{\infty} T^k,
\]
where the (Neumann series) series converges with respect to the norm of \( B(X) \).

Proof. We have already shown that \( \|T\| < 1. \) Clearly \( \sum_{k=0}^{\infty} \|T^k\| < \infty \) (By Root test). Since \( B(X) \) is complete, then by the completeness the series \( S = \sum_{k=0}^{\infty} T^k \) is convergent. Now,
\[
(I - T)S = (I - T) \sum_{k=0}^{\infty} T^k = \sum_{k=0}^{\infty} (T^k - T^{k+1}) = (I - T) + (T - T^2) + ... = I
\]
Similarly, we can show that \( S(I - T) = I \). Which means the operator \( I - T \) is a bijective operator on \( X \) and that its inverse is given by the Neumann series \( S \) defined above. Again, since for every \( x \in X \) we have \( \|Sx\| \leq (\sum_{k=0}^{\infty} \|T^k\|) \|x\| \) and \( (I - T)^{-1} \) is bounded on \( X \) with norm less or equal to \( \sum_{k=0}^{\infty} \|T^k\| \).
Now, from above theorem we can say that if \( \|R_c(T)\| \|S\| < 1 \) then the series obtained from iterating Resolvent identity converges with respect to the operator norm \( \|\cdot\| \) on \( B(X) \) and gives

\[
R_c(T + S) = R_c(T) + R_c(T + S)SR_c(T)
\]

\[
(1 - R_c(T)S)R_c(T + S) = R_c(T)
\]

\[
R_c(T + S) = R_c(T)(1 - SR_c(T))^{-1}
\]

\[
= R_c(T) + \sum_{j=1}^{\infty} R_c(T)(SR_c(T))^j
\]

\[
= \sum_{j=0}^{\infty} S^j(R_c(T))^{j+1}
\]

In the special case \( S = (c - w)I \) with

\[
|c - w| < \|R_c(T)\|^{-1}
\]

This results in

\[
R_w(T) = R_c(T + (c - w)I)
\]

\[
= \sum_{j=0}^{\infty} (c - w)^j(R_c(T))^{j+1}.
\]

Now we have the spectral radius \( r = \lim_{n \to \infty} \sup \|T^n\|^{\frac{1}{n}} \) and we have from the lecture note on February 2, 2017 the resolvent set of \( T \) is \( \rho(T) = \{c \in \mathbb{C} : T - CI \text{ is invertible}\} \) and \( \rho(T) = \mathbb{C}\setminus\sigma(T) \) we can see from the sketch below.

![Figure 1: Sketch](image)

**3.11 Theorem.** For \( T \in B(x) \),

\[
\max\{|z| : z \in \sigma(T)\} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}
\]
Proof. Let \( r = \limsup_{n \to \infty} \|T^n\|^{\frac{1}{n}} \) then we show there is \( z \in \sigma(T), |z| = r. \)

If \( r = 0 \) then from Gelfand theorem (from the lecture note on February 23, 2017) we have

\[
\phi \neq \sigma(T) \subset B_0(0) \Rightarrow \sigma(T) = \{0\}.
\]

Next, consider \( r > 0 \). Assume \( \sigma(T) \cap \{z : |z| = r\} = \phi \). Then,

\[
\max\{|z| : z \in \sigma(T)\} < r.
\]

Take \( R > 0 \) such that

\[
r(T) = \max\{|z| : z \in \sigma(T)\} < R < r
\]

Then,

\[
\sigma(T) \subset \overline{B_{r(T)}(0)}
\]

By the series computation from the proof of Warm of theorem above (Theorem 6.1.2), for \( f \in B(x)^t \) then \( g(z) = f((T - zI)^{-1}) \) defines a holomorphic function \( g \) on \( \rho(T) \supset \{z : |z| > r(T)\} \) with

\[
g(z) = -\sum_{n=0}^{\infty} f(T^n)z^{-(n+1)}
\]

The domain of analyticity includes \( \{z \in : |z| = R\} \) so

\[
\sup_{n \geq 0} \frac{|f(T^n)|}{R^{n+1}} < \infty. \text{ (since the series is convergent)}
\]
This is true for any $f \in B(x)'$ with $\|f\| \leq 1$. So from uniform boundedness (from the lecture note January 31, 2017).

$$c = \sup_{n \geq 0} \left\| \frac{T^n}{R^{n+1}} \right\| < \infty \text{ or } \|T^n\| \leq cR^{n+1}$$

And thus,

$$\|T^n\|^{\frac{1}{n}} \leq c^{\frac{1}{n}} R^{1 + \frac{1}{n}}$$

So $\lim\sup_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r \leq R$, which contradicts with our assumption $R < r$.

Next, we show $r = \inf\{\|T^n\|^{\frac{1}{n}}\}$.

Let $n, m \in \mathbb{N}$ then $n = qm + k, \ k \in \{0, 1, 2, ..., m - 1\}$ and by the fundamental norm inequality for operator norm we have,

$$\|T^n\| \leq \|T^m\|^q \|T\|^k$$

so

$$\|T^n\|^{\frac{1}{n}} \leq \|T^m\|^q \|T\|^k$$

Fixing $m$ and letting $n \to \infty$, by $n = qm + k \to 1 = \frac{qm}{n} + \frac{k}{n}$ we get $\frac{k}{n} \to \infty, \ \frac{q}{n} \to \frac{1}{m}$ so,

$$r = \lim_{n \to \infty} \sup \|T^n\|^{\frac{1}{n}} \leq \|T^m\|^{\frac{1}{m}}$$

Then taking the infimum over $m \in \mathbb{N}$, we get

$$r \leq \inf_{m=1} \|T^m\|^{\frac{1}{m}}$$

$$\leq \lim_{m \to \infty} \inf \|T^m\|^{\frac{1}{m}}$$

$$\leq \lim_{m \to \infty} \sup \|T^m\|^{\frac{1}{m}} = r$$

Hence the limit exits and equality holds throughout. \hfill \square

3.12 Example. Let $T : l^1 \to l^1$ be defined by $T(x_1, x_2, ...) = (x_2, x_3, ...)$ then

$$\|T(x_1, x_2, ..., )\|_1 = \|(x_1, x_2, ..., )\|_1 = \sum_{j=2}^{\infty} |x_j| \leq \sum_{j=2}^{\infty} |x_j| = \|x\|_1$$

Hence $T$ is contraction.

Also, setting $x_1 = 0$ shows $\|T\| = 1$, then by above theorem, $\sigma(T) \subset \overline{B}_r(T)(0)$, and

$$r = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq \|T\| = 1$$

We show $r(T) = 1$, we see that if $|z| < 1$, then

$$T(1, z, z^2, ...) = (z, z^2, z^3, ...) = z(1, z, z^2, ...)$$

So $z \in \sigma(T) \cong ker(T - zI)$ and hence

$$B_1(0) \subset \sigma(T) \subset \overline{B}_1(0).$$

We conclude by closeness of $\sigma(T), \sigma(T) = \overline{B}_1(0)$. 4
3.13 Example. It is possible for $T$ to be injective $\sigma(T) = \{0\}$. Let $C([0, 1])$ is equipped with $\|\cdot\|_\infty$, and let $T$ be given by

$$ T : C([0, 1]) \rightarrow C([0, 1]) \text{ be } Tf(x) = \int_0^x f(t)dt $$

Then,

$$ \|Tf\|_\infty \leq \sup_{x \in [0, 1]} \int_0^x |f(t)|dt $$

$$ \leq \int_0^1 \|f\|_\infty dt $$

$$ = \|f\|_\infty $$

So $T$ is contraction.

For $f = 1$, $\|Tf\|_\infty = 1$, $\|T\| = 1$.

Next, we see

$$ |T^nf(x)| \leq \|f\|_\infty \frac{x^n}{n!} $$

Let $n = 1$, then $|Tf(x)| \leq \|f\|_\infty x$. Assuming, inequality holds for $n \in \mathbb{N}$, then

$$ |(T^{n+1}f)x| = \int_0^x T^n f(t)dt $$

$$ \leq \int_0^x |T^n f(t)|dt $$

$$ \leq \int_0^x |T^n f(t)|dt \text{ (by induction assumption)} $$

$$ = \|f\|_\infty \frac{x^{n+1}}{(n+1)!} $$

Consequently,

$$ \|T^n f\|_\infty = \|f\|_\infty \frac{1}{n!} $$

And for $f = 1$, we get equality. Thus,

$$ \|T^n\|_{\frac{1}{n}} = \left(\frac{1}{n!}\right)^{\frac{1}{n}} n \rightarrow 0. $$

So, $r(T) = 0$ Hence, $T$ is injective, but the spectrum is the same as that of the zero map this means $\sigma(T) = 0$. 