Last Time: Spectrum of multiplication operators and complemented subspaces.

3.B More Examples of Complemented Subspaces

3. Every closed subspace $E$ in a Hilbert space is complemented by $E^\perp$.

4. If
   \[ H^p = \text{span}\{e^{2\pi inx}\}_{n=0}^{\infty} \in L^p([0,1]), \text{ } 1 < p < \infty, \]
   then $H^p$ is complemented by
   \[ \text{span}\{e^{2\pi inx}\}_{n<0} \]

It will be shown that projections do in fact provide an equivalent formulation of complemented subspaces.

3.20 Definition. An operator $P \in B(X)$ is a projection if $P^2 = P$.

3.21 Claim. $\ker P = \text{ran}(I - P)$.

Proof. If $Px = 0$, then $x = x - Px = (I - P)x$, and so if $x \in \ker P$, then $x \in \text{ran}(I-P)$. Conversely, if $y = (I - P)x$, then
\[ Py = P(I - P)x = (P - P^2)x = Px - Px = 0. \]
So if $y \in \text{ran}(I - P)$, then $y \in \ker P$. \hfill \Box

The following two theorems are the equivalent formulation of complemented subspaces provided by projections; the first theorem is the ”easy” direction of such equivalence and part of the second is its converse.

3.22 Theorem. If $P$ is a projection, then $\text{ran} P$ is closed and complemented by $\ker P$. 
Proof. Let \( Q = I - P \). Then \( Q \) is a projection, \( \ker Q = \text{ran}(I - Q) \) by the previous claim, and \( \text{ran}(I - Q) = \text{ran}P \). Moreover, \( \text{ran}P \) is complemented by \( \ker P = \text{ran}(I - P) \) because if \( x \in \text{ran}P = \ker(I-P) \) and \( x \in \ker \), then
\[
0 = (I - P)x = x.
\]
Finally, for any \( z \in X \), \( z \) is given by
\[
z = Pz + (I - P)z
\]
where \( Pz \in \text{ran}P \) and \( (I - P)z \in \ker P \). \( \square \)

**3.23 Theorem.** A closed subspace \( E \) of \( X \) is complemented if and only if there is a projection \( P \in B(x) \) such that \( P^2 = P \) with \( E = \text{ran}P \).

Proof. If there is a projection, then by the theorem above, \( E = \text{ran}P \) is complemented. Conversely, let \( F \) be a complementary subspace to \( E \). If \( z \in X \), with \( x \in E \) and \( y \in F \) being unique, one can write \( z = x + y \). Let \( Pz = x \), then by uniqueness, this is a well-defined linear map. Also, \( \text{ran}P = E \) because if \( x \in E \) and \( 0 \in F \), then
\[
x = x + 0 \Rightarrow Px = x.
\]
Moreover,
\[
P^2z = P(Pz) = Px = x = Pz, \quad Pz \in \text{ran}P.
\]
Hence, \( P^2 = P \).

To show \( P \in B(X) \), consider the graph of \( P \in X \oplus X \) with norm \( ||(z, x)|| = ||z|| + ||x|| \), and let \( (z_n, x_n) \to (z, x) \). Then \( z_n = x_n + y_n \) where for each \( n \in \mathbb{N} \), \( x_n \in E \) and \( y_n \in F \). So, \( Pz_n = x_n \to x \in E \) by \( \overline{E} = E \). Consequently, \( y_n = z_n - x_n \to z - x \in F \) since \( F \) is closed. Thus, \( z = x + (z - x) \) and \( Pz = x \). Therefore, the limit is in the graph of \( P \) and hence \( P \) has a closed graph. Using the Closed Graph theorem, \( P \) is bounded. \( \square \)

Complemented subspaces can also be used to study weak forms of invertiblity. Given Banach spaces \( X \) and \( Y \), if \( T \in B(X, Y) \), then \( T \) is said to be **left-invertible** if there is \( S \in B(Y, X) \) such that \( ST = I_X \).

**3.24 Theorem.** Let \( X \) and \( Y \) be Banach spaces and \( T \in B(X, Y) \). Then \( T \) is left-invertible if and only if \( T \) is injective and \( \text{ran}T \) is closed and complemented.

Proof. If \( T \) is injective and \( \text{ran}T \) is closed and complemented, then taking \( P \) as the projection onto \( \text{ran}T \),
\[
T_0 = P \circ T : X \to \text{ran}T
\]
is a projection onto a Banach space, so it is invertible by the open mapping theorem [W. Rudin, Theorem 2.11, (1)]. Hence, if \( S = T_0^{-1}P \), then \( ST = T_0^{-1}PT = I_X \) where \( PT = T_0 \). Therefore, \( T \) is left-invertible. On the other hand, if \( S \in B(Y, X) \) is such that \( ST = I_X \), then \( T \) is injective and
\[
(TS)^2 = (TS)(TS) = T(ST)S = TS.
\]
So, $TS$ is a projection with $\text{ran}(TS) \subset \text{ran}T$, but

$$\text{ran}T = \text{ran}(TST) \subset \text{ran}(TS)$$

and hence $\text{ran}T = \text{ran}(TS)$. Therefore, $\text{ran}T$ is the range of the projection that is closed and complemented. \hfill \square

A second look at the result on ergodicity; recall that to show $A_n x = \frac{1}{n} \sum_{j=1}^{n} T^j x \rightarrow T x$, one must assume power boundedness, $\sup_{n \in \mathbb{N}} ||T^n|| < \infty$.

3.25 Proposition. If $T \in B(X)$ is power bounded, then $r(T) \leq 1$.

Proof. From $||T^n|| \leq C$, one gets

$$\lim_{n \to \infty} ||T^n||^{\frac{1}{n}} \leq \lim_{n \to \infty} C^{\frac{1}{n}} = 1.$$  

It was shown that $\overline{T} x = \lim_{n \to \infty} A_n x = y$ with $T y = y$, equivalently, $y \in \ker(I - T)$. Conversely, if $y \in \ker(I - T)$, then $A_n y = y$ for each $n \in \mathbb{N}$, so $\overline{T} y = y$. Therefore, $\overline{T}(X) = \ker(I - T)$. \hfill \square

One can also characterize the kernel of $\overline{T}$. From the statement on complementary projection, the $\ker\overline{T} = \text{ran}(I - \overline{T})$.

3.26 Corollary. If $T$ is as above, then the spaces $E = \ker T$ and $F = \text{ran} T$ are complementary and $F = \ker(I - T)$.

References