Last Time

- Properties of compact operators
- Approximating compact by finite-rank operators
- Characterization by complete continuity

3.36 Theorem. Let $H$ be a separable Hilbert space and $T \in B(H)$ be compact, then $T$ is the limit of a sequence of finite rank operators.

Proof. If $T$ is finite rank, then nothing to show. Otherwise, take $\text{ran}(T)$, find orthonormal basis $\{e_1, e_2, \ldots \}$ for $H$ and let $H_n := \text{span}\{e_k\}_{k=1}^n$, then the orthogonal projections $P_n$ defined by

$$P_n x = \sum_{j=1}^n \langle x, e_j \rangle e_j.$$  

By Bessel’s inequality, $\sum_{j=1}^n |\langle x, e_j \rangle|^2 \leq \|x\|^2$ and cases of equality, we have that

$$\|P_n\| = 1 = \|I - P_n\|.$$  

Moreover, for $1 \leq m \leq n$, $P_n P_m = P_m$ and $(I - P_n)(I - P_m) = I - P_n$. We know $T_n := P_n T$ is finite rank, consequently,

$$\|T - P_n T\| = \|(I - P_n)T\|$$

$$= \|(I - P_n)(I - P_m)T\|$$

$$\leq \|(I - P_m)T\| = \|T - P_m T\|.$$  

We conclude $(\|T - P_n T\|)_{n \in \mathbb{N}}$ is non-increasing.

If $\|T - P_n T\| \to 0$, then the statement is proved, so assume, for a contradiction, the limit of the sequence of norms is non-zero, $\epsilon > 0$. Then, by definition of operator norm, for each $n \in \mathbb{N}$ there is $x_n$ with $\|x_n\| = 1$ and $\|(I - P_n)Tx_n\| > \frac{\epsilon}{2}$. By reflexivity $(x_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$, $x_{n_k} \xrightarrow{w} x$. 

From the characterization of compactness by complete continuity, $Tx_n \to u = Tx$.

By $u \in \text{ran}(T)$, $\|P_n u - u\| \to 0$ as $n \to \infty$, then

$$\frac{\epsilon}{2} < \|(I - P_n)Tx_n\| \leq \|(I - P_n)(Tx_n - u)\| + \|(I - P_n)u\| \leq \|Tx_n - u\| + \|u - P_n u\| \to 0$$

since $(I - P_n)$ always has norm 1, and both the terms on the RHS converge to zero, which is our desired contradiction.

Hence, $\|T - P_n T\| \to 0$, so $T$ is the limit of sequence $(P_n T)_{n \in \mathbb{N}}$ of finite rank operators.

3.37 Definition. A Banach space $Y$ has the approximation property if for each Banach space $X$, every compact $T \in B(X,Y)$ is the limit of a sequence of finite rank operators.

Grothendieck proved that $Y$ has this property if and only if for every compact subset $W$ of $Y$, and every $\epsilon > 0$, there is a finite rank operator $T \in B(Y)$ such that for all $y \in W$, $\|Ty - y\| < \epsilon$. Next to separable Hilbert spaces, $c_0$ and $l^p$, $1 \leq p < \infty$ have this property. However, not every reflexive separable Banach space has this property. [Enflo, P. A counterexample to the approximation property in Banach spaces. Acta Math. 130, 309-317(1973)]. It was later shown by Szankowski that there exist closed linear subspaces of $l^p$ (with $1 \leq p < \infty$ and $p \neq 2$) and of $c_0$ that do not have the approximation property. [A. Szankowski, Subspaces without the approximation property. Israel J. Math. 30 (1978), 123-129].

Next, we examine properties of compact operators that resemble conclusions drawn from the Jordan form in finite dimensions.

We begin with a lemma.

3.38 Lemma. If $X$ is Banach space, $T \in B(X)$ is compact and $c \neq 0$, then $N = \ker(T - cI)$ is finite dimensional and $M = \text{ran}(T - cI)$ is closed and of finite codimension.

3.39 Remark. Note $N$ and $M$ may not be complementary, e.g. on $\mathbb{R}^2$, if

$$T - cI = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then $T - cI$ has kernel equal to its range.

Proof. For $N$, note $N$ is a closed subspace, so $T|_N$ is compact, but by definition of $N$, $T|_N = cI_N$, so this is invertible, hence $N$ is finite dimensional.

Regarding $M$, note $M^\perp = \ker(T^t - cI)$ which is finite dimensional. Consider a complementary subspace $Z$ of $N$. Let $S = (T - cI)|_Z$, then $S$ is injective by $Z \cap N = \{0\}$.

From $\text{ran}(S) = M$, in order to show $M$ is closed, we only need to establish $S$ is norm-bounded below.

If not, then there is $(Z_n)_{n \in \mathbb{N}}$, $z_n \in Z$, $\|z_n\| = 1$, with $S z_n \to 0$. By compactness of $S$, we can choose a subsequence such that $z_{n_k} \xrightarrow{w} v$ and $S z_{n_k} \to w = S v$. 

\[\]
We need to show $v \neq 0$ in order to contradict injectivity.

On $Z$, we have $(T - S)|_Z = cI|_Z$, so

$$\frac{1}{c}(T - S)z_{n_k} = z_{n_k} \rightarrow v$$

but the convergence is also in norm by $S$, $T$ is compact, so we have $\|v\| = 1$.  \qed