Functional Analysis II, Math 7321
Lecture Notes from March 23, 2017

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Last Time

- Properties of compact operators
- Approximating compact by finite-rank operators
- Characterization by complete continuity

3.36 Theorem. *Let H be a separable Hilbert space and $T \in B(H)$ be compact, then T is the limit of a sequence of finite rank operators.*

Proof. If T is finite rank, then nothing to show. Otherwise, take $\overline{ran(T)}$, find orthonormal basis $\{e_1, e_2, \dots\}$ for H and let $H_n := span\{e_k\}_{k=1}^n$, then the orthogonal projections P_n defined by

$$P_n x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j.$$

By Bessel's inequality, $\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \leqslant \|x\|^2$ and cases of equality, we have that

$$\|P_n\| = 1 = \|I - P_n\|.$$

Moreover, for $1 \leqslant m \leqslant n$, $P_n P_m = P_m$ and $(I - P_n)(I - P_m) = I - P_n$.
We know $T_n := P_n T$ is finite rank, consequently,

$$\begin{aligned} \|T - P_n T\| &= \|(I - P_n)T\| \\ &= \|(I - P_n)(I - P_m)T\| \\ &\leqslant \|(I - P_m)T\| = \|T - P_m T\|. \end{aligned}$$

We conclude $(\|T - P_n T\|)_{n \in \mathbb{N}}$ is non-increasing.
If $\|T - P_n T\| \to 0$, then the statement is proved, so assume, for a contradiction, the limit of the sequence of norms is non-zero, $\epsilon > 0$. Then, by definition of operator norm, for each $n \in \mathbb{N}$ there is x_n with $\|x_n\| = 1$ and $\|(I - P_n)T x_n\| > \frac{\epsilon}{2}$. By reflexivity $(x_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$, $x_{n_k} \xrightarrow{w} x$.

From the characterization of compactness by complete continuity, $Tx_{n_k} \to u = Tx$.
By $u \in ran(T)$, $\|P_n u - u\| \to 0$ as $n \to \infty$, then

$$\begin{aligned} \frac{\epsilon}{2} &< \|(I - P_{n_k})Tx_{n_k}\| \\ &\leqslant \|(I - P_{n_k})(Tx_{n_k} - u)\| + \|(I - P_{n_k})u\| \\ &\leqslant \|Tx_{n_k} - u\| + \|u - P_{n_k}u\| \to 0 \end{aligned}$$

since $(I - P_{n_k})$ always has norm 1, and both the terms on the RHS converge to zero, which is our desired contradiction.
Hence, $\|T - P_n T\| \to 0$, so T is the limit of sequence $(P_n T)_{n\in\mathbb{N}}$ of finite rank operators. □

3.37 Definition. A Banach space Y has the approximation property if for each Banach space X, every compact $T \in B(X, Y)$ is the limit of a sequence of finite rank operators.

Grothendieck proved that Y has this property if and only if for every compact subset W of Y, and every $\epsilon > 0$, there is a finite rank operator $T \in B(Y)$ such that for all $y \in W$, $\|Ty - y\| < \epsilon$. Next to separable Hilbert spaces, c_0 and l_p, $1 \leqslant p < \infty$ have this property. However, not every reflexive separable Banach space has this property. [Enflo, P. A counterexample to the approximation property in Banach spaces. Acta Math. 130, 309-317(1973)]. It was later shown by Szankowski that there exist closed linear subspaces of l^p(with $1 \leqslant p < \infty$ and $p \neq 2$) and of c_0 that do not have the approximation property. [A. Szankowski, Subspaces without the approximation property. Israel J. Math. 30 (1978), 123129].
Next, we examine properties of compact operators that resemble conclusions drawn from the Jordan form in finite dimensions.
We begin with a lemma.

3.38 Lemma. *If X is Banach space, $T \in B(X)$ is compact and $c \neq 0$, then $N = ker(T - cI)$ is finite dimensional and $M = ran(T - cI)$ is closed and of finite codimension.*

3.39 Remark. Note N and M may not be complementary, e.g. on $\mathbb{R}^2$, if

$$T - cI = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then $T - cI$ has kernel equal to its range.

Proof. For N, note N is a closed subspace, so $T|_N$ is compact, but by definition of N, $T|_N = cI_N$, so this is invertible, hence N is finite dimensional.
Regarding M, note $M^\perp = ker(T' - cI)$ which is finite dimensional. Consider a complementary subspace Z of N. Let $S = (T - cI)|_Z$, then S is injective by $Z \cap N = \{0\}$.
From $ran(S) = M$, in order to show M is closed, we only need to establish S is norm-bounded below.
If not, then there is $(Z_n)_{n\in\mathbb{N}}$, $z_n \in Z$, $\|z_n\| = 1$, with $Sz_n \to 0$. By compactness of S, we can choose a subsequence such that $z_{n_k} \xrightarrow{w} v$ and $Sz_{n_k} \to w = Sv$.

We need to show $v \neq 0$ in order to contradict injectivity.
On Z, we have $(T-S)|_Z = cI|_Z$, so

$$\frac{1}{c}(T-S)z_{n_k} = z_{n_K} \to v$$

but the convergence is also in norm by S, T is compact, so we have $\|v\| = 1$.

□