3.40 Remark. Recall that we defined the compactness of $T \in B(X,Y)$ in terms of $\overline{T(B_1(0))}$ [Notes, March 9]. However, we could have equivalently defined $T$ to be compact if $\overline{T(B_1(0))}$ is compact in $Y$; we showed that compactness of the closure of the image of the unit ball is equivalent to the condition that any bounded sequence has a strongly convergent subsequence, using a bound of 1, but we may set the bound to $1 + \epsilon$ to demonstrate the equivalence of the stronger formulation. Since $T(B_1(0)) \supset T(B_1(0))$, it may be more difficult to prove compactness in terms of the larger set.

Warm-up

We note that we may have incidentally used the following result without having shown it.

3.41 Claim. If $T \in B(X,Y)$ is compact, then $T'$ is compact.

Proof. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $Y'$ with $\|y_n\| \leq 1$ for all $n \in \mathbb{N}$. Considering $\overline{B_1(0)} \subset X$, $T$ compact implies $W := \overline{T(B_1(0))}$ is compact in $Y$. Thus, we may apply Ascoli’s Theorem [A5, (1)] once establishing that $(y_n)_{n \in \mathbb{N}}$ is an equicontinuous family on $W$.

Given $x, z \in W$, we have $|y_n(x) - y_n(z)| = |y_n(x - z)|$ by linearity, which can only increase by taking the norm of $y_n$, so $|y_n(x - z)| \leq \|y_n\| \|x - z\| \leq \|x - z\|$. Thus, $(y_n)_{n \in \mathbb{N}}$ is a uniformly bounded equicontinuous family in $C(T(B_1(0)))$. The corollary to Ascoli’s Theorem provides a uniformly convergent subsequence $(y_{n_k})_{k \in \mathbb{N}}$ in $C(T(B_1(0)))$. Since $T$ is continuous and $T' y_{n_k} = y_{n_k} \circ T$, the sequence $(T' y_{n_k})_{k \in \mathbb{N}}$ is uniformly convergent in $C(\overline{B_1(0)})$. By the definition of the norm in $X'$, this means $(T' y_{n_k})_{k \in \mathbb{N}}$ is convergent in norm. Thus, we have found a norm convergent subsequence of $(T' y_{n_k})_{k \in \mathbb{N}}$, and $T'$ is compact.

3.42 Theorem (Riesz-Fredholm). Let $T \in B(X)$ be compact, $c \neq 0$, $N_j = \ker(T - cI)^j$ and $M_j = \text{ran}(T - cI)^j$; then the following hold:

(a) $N_1 \subset N_2 \subset \cdots$, and there exists $k_N \in \mathbb{N}$ such that $N_{k_N - 1} \subset N_{k_N} = N_{k_N + j}$ for all $j \in \mathbb{N}$, and all $N_j$ are invariant under $T$ and of finite dimension;

(b) $M_1 \supset M_2 \supset \cdots$, and there exists $k_M \in \mathbb{N}$ such that $M_{k_M - 1} \supset M_{k_M} = M_{k_M + j}$ for all $j \in \mathbb{N}$, and all $M_j$ are invariant under $T$, closed and of finite codimension;
(c) \( k_N = k_M =: k \) and \( M \) and \( N_k \) are complementary (closed) subspaces. In addition, \((T - cI)|_{M_k}\) is invertible in \( B(M_k) \), and \((T - cI)|_{N_k}\) is nilpotent in \( B(N_k) \) with index \( k \), meaning \((T - cI)^{k-1}|_{N_k} \neq 0 \) and \((T - cI)^k|_{N_k} = 0\).

(d) \( \text{dim ker}(T - cI) = \text{codim ran}(T - cI) \)

**Proof.** (a) The inclusion is clear from composition, and we now show that each \( N_j \) is invariant under \( T \): fixing \( j \) and taking \( x \in N_j \), we note that \( T \) commutes with each of the terms of \((T - cI)^j\). Thus \((T - cI)^jTx = T(T - cI)^jx\), which equals 0 since \( x \in N_j \), and \( Tx \in N_j \).

To show the existence of \( k_N \) as in the statement of the theorem, we demonstrate that it is enough to show that \( N_k = N_{k+1} \) for some \( k \in \mathbb{N} \): Suppose such a \( k \) exists, and take \( k_N \) to be the least such index. Then for \( x \in N_{kN+2} \), we have that \((T - cI)x \in N_{kN+1} = N_{kN} \), which implies \( x \in N_{kN+1} \) and \( N_{kN+1} = N_{kN+2} \). An induction argument yields that \( N_{kN} = N_{kN+j} \) for all \( j \in \mathbb{N} \). Thus, \( k_N \) exists if \( N_k = N_{k+1} \) for some \( k \).

In pursuit of a contradiction, suppose that for each \( j \in \mathbb{N} \), \( N_j \subseteq N_{j+1} \). We note that each \( N_j \) is closed, as the kernel of a bounded operator, and we produce a sequence \((x_j)_{j \in \mathbb{N}} \) by choosing \( x_j \in N_j \setminus N_{j-1} \) for each \( j \) such that \( x_j + N_{j-1} \in N_j \) and \( \|x_j + N_{j-1}\| = 1 \).

For \( j > 1 \), we choose \( y_j \in N_{j-1} \) such that \( \|x_j + y_j\| \leq 2 \); the existence of such points follows from the definition of the quotient norm an infimum of distances. Setting \( x'_j := x_j + y_j \), we have that \( x'_j + N_j = x_j + N_j \), \( \|x'_j\| \leq 2 \), and \( \|x'_j + N_{j-1}\| = 1 \) for each \( j \in \mathbb{N} \setminus \{1\} \).

Consider, for \( i < j \in \mathbb{N} \), the differences \( Tx'_j - Tx'_i \), which we may artificially write as

\[
Tx'_j - Tx'_i = cx'_j + (Tx'_j - cx'_j) - Tx'_i \\
\quad \in cx'_j + N_{j-1} + N_i, \quad \text{(since } Tx'_j - cx'_j = (T - cI)x'_j \text{ and } (T - cI)^j(T - cI)x'_j = 0 \text{)}
\]

\[
= cx'_j + N_{j-1} \\
= c(x'_j + N_{j-1}) \\
= c(x_j + N_{j-1}).
\]

Since \( \|x_j + N_{j-1}\| = 1 \), it follows that \( \|Tx'_j - Tx'_i\| \geq \|c(x_j + N_{j-1})\| = |c| > 0 \) for all \( i < j \in \mathbb{N} \). Thus \((Tx'_j)_{j \in \mathbb{N}} \) has no convergent subsequence, which is absurd in the light of \( T \)'s compactness. We conclude that our supposition was false, and the existence of \( k_N \) is shown.

Finally, since \( T \) is compact, so is

\[
S := \sum_{m=1}^{i} \binom{i}{k} (-c)^{i-m} T^m \\
=(T - cI)^i - (-c)^i I;
\]

restricting \( S \) to \( N_i \) means \((T - cI)^i = 0\), and we have \( S|_{N_i} \) in \( B(N_i) \) is equal to \((T - cI)^i\). The compactness of the identity operator means that \( \text{dim } N_i < \infty \). \( \square \)
References