Functional Analysis II, Math 7321
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3.C Normal operators

3.53 Definition. Let H be a Hilbert space over $\mathbb{C}$ and let $T \in B(H)$.

1. T is normal if $T^*T = TT^*$, or equivalently if $\langle T^*Tx, x\rangle = \langle TT^*x, x\rangle$ for all $x \in H$, or equivalently if $\|Tx\|^2 = \|T^*x\|^2$ for all $x \in H$.

2. T is self-adjoint if $T = T^*$, or equivalently if

$$\langle Tx, x\rangle = \langle T^*x, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle},$$

for all $x \in H$.

3.54 Theorem. *Let H be a Hilbert space over $\mathbb{C}$ and let $T \in B(H)$ be normal.*

(a) For every $c \in \mathbb{C}$, $T - cI$ is normal, and if T is invertible, T^{-1} is also normal.

(b) If $a \neq b$ and $x, y \in H$ with $Tx = ax$, $Ty = by$, then $\langle x, y\rangle = 0$.

(c) $r(T) = \|T\|$.

Proof. (a) We have

$$\begin{aligned}(T - cI)(T - cI)^* &= (T - cI)(T^* - \overline{c}I)\\ &= TT^* - cT^* - \overline{c}T + |c|^2 I\\ &= T^*T - cT^* - \overline{c}T + |c|^2 I\\ &= (T^* - \overline{c}I)(T - cI)\\ &= (T - cI)^*(T - cI),\end{aligned}$$

where we used the fact that $TT^* = T^*T$. Hence $T - cI$ is normal. Next, if T is invertible, $(T^{-1})^* = (T^*)^{-1}$, and so

$$\begin{aligned}T^{-1}(T^{-1})^* &= T^{-1}(T^*)^{-1} = (T^*T)^{-1}\\ &= (TT^*)^{-1} = (T^*)^{-1}T^{-1}\\ &= (T^{-1})^*T^{-1}\end{aligned}$$

that T^{-1} is normal.
(b) Starting from the fact that $Ty = by$ if and only if $T^*y = \bar{b}y$ we have

$$0 = \|(T - bI)y\| = \|(T - bI)^*y\| = \|(T^* - \bar{b}I)y\|,$$

and so y is an eigenvector of T^* with corresponding eigenvalue $\bar{b}$. Next,

$$\begin{aligned} a\langle x, y\rangle &= \langle Tx, y\rangle = \langle x, T^*y\rangle \\ &= \langle x, \bar{b}y\rangle = b\langle x, y\rangle, \end{aligned}$$

and so $(a - b)\langle x, y\rangle = 0$. But, $a \neq b$, so $\langle x, y\rangle = 0$.
(c) We have

$$\begin{aligned} \|T^2\| &= \|(T^*)^2T^2\|^{\frac{1}{2}} \\ &= \|(T^*T)^*(T^*T)\|^{\frac{1}{2}} \\ &= \|T^*T\| \\ &= \|T\|^2, \end{aligned}$$

where we have used the C^*-identity and normality. So by iterating the above process we get $\|T^{2^n}\| = \|T\|^{2^n}$ and thus

$$r(T) = \lim_{n\to\infty} \|T^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|T\| = \|T\|,$$

which completes the proof. □

3.55 Definition. Let H be a Hilbert space over $\mathbb{C}$ and $T \in B(H)$. Also let M be a subspace of H. We say M reduces T if $T(M) \subset M$ and $T(M^\perp) \subset M^\perp$.

3.56 Lemma. *Let H be a Hilbert space and $T \in B(H)$. Also let M be a closed subspace of H and P an orthogonal projection with range M. Then,*

(a) $T(M) \subset M$ if and only if $PTP = TP$, or equivalently if and only if $T^*(M^\perp) \subset M^\perp$. Also $T(M^\perp) \subset M^\perp$ if and only if $PTP = PT$, or equivalently if and only if $T^*(M) \subset M$.

(b) M reduces T if and only if $PT = TP$, or equivalently if and only if M reduces T^*.

Proof. Let $x \in H$ and write $x = y_1 + y_2$ and $Ty_1 = z_1 + z_2$ with $y_1, z_1 \in M$ and $y_2, z_2 \in M^\perp$. We have

$$PTPx = PTy_1 = z_1 \in M$$

and $TPx = Ty_1 = z_1 + z_2$. Thus, $PTP = TP$ if and only if $TPx = z_1 \in M$. Hence, $TPx \in M$ for all $x \in H$ if and only if $T(M) \subset M$.

Next, $I - P$ is the orthogonal projection onto $M^\perp$, so repeating the previous argument implies $T^*(M^\perp) \subset M^\perp$ if and only if $(I - P)T^*(I - P) = T^*(I - P)$, or equivalenty if and only if $PT^*P = PT^*$, or equivalently, by taking adjoints on both sides, if and only if $PTP = TP$.
(B) By definition, M reduces T if $T(M) \subset M$ and $T(M^\perp) \subset M^\perp$, which by (a) is true if and only if $PTP = TP$ and $PTP = PT$, or equivalently if and only if $TP = PT$. □

Nest, we deduce properties of normal operators.

3.57 Theorem. *Let H be a Hilbert space over $\mathbb{C}$ and $T \in B(H)$ be normal. Also let M be a closed subspace of H. Then,*

(a) for every $c \in \mathbb{C}$, $ker(T - cI)$ reduces T and T^*.

(b) If M reduces T, then $T|_M$ and $T|_{M^\perp}$ are normal operators on M and $M^\perp$, respectively, and

$$\|T\| = \max\{\|T|_M\|, \|T|_{M^\perp}\|\}.$$

Proof. (a) We take $x \in ker(T - cI)$ and notice that

$$(T - cI)Tx = T(T - cI)x = 0,$$

since T and $T - cI$ commute. Thus, $Tx \in ker(T - cI)$. Similarly,

$$(T - cI)T^*x = T^*(T - cI)x = 0,$$

since T is normal. So $T^*x \in ker(T - cI)$. Thus, $ker(T - cI)$ reduces T and T^*.
(b) Note that $(T|_M)^* = T^*|_M$ and

$$\begin{aligned} T|_M(T|_M)^* &= T|_M T^*|_M = TT^*|_M \\ &= T^*T|_M = T^*|_M T|_M \\ &= (T|_M)^*(T|_M), \end{aligned}$$

by normality. So $T|_M$ and $T^*|_M$ are normal. Similarly, $T|_{M^\perp}$ and $T^*|_{M^\perp}$ are normal.

Next, let $a := \max\{\|T|_M\|, \|T|_{M^\perp}\|\}$. Then, since $\|T|_M\| \leq \|T\|$ and $\|T|_{M^\perp}\| \leq \|T\|$, we have $a \leq \|T\|$. On the other hand and for $x = y + z$ with $y \in M$, $z \in M^\perp$, we have

$$\|x\|^2 = \|y\|^2 + \|z\|^2,$$

by the Pythagorean theorem. Since M reduces T, $Ty \in M$ and $Tz \in M^\perp$ and so the Pythagorean theorem once again gives

$$\begin{aligned} \|Tx\|^2 &= \|Ty\|^2 + \|Tz\|^2 \\ &\leq \|T|_M\|^2\|y\|^2 + \|T|_{M^\perp}\|^2\|z\|^2 \\ &\leq \max\{\|T|_M\|^2, \|T|^2_{M^\perp}\|^2\}(\|y\|^2 + \|z\|^2) \\ &= \max\{\|T|_M\|^2, \|T|^2_{M^\perp}\|^2\}\|x\|^2, \end{aligned}$$

which means $\|T\| \leq a$. □

3.58 Theorem. *Let H be a Hilbert space over $\mathbb{C}$ and let $T \in B(H)$ be compact and normal. For any $c \in \sigma(T)$, let P_c be the orthogonal projection onto to $H_c = ker(T - cI)$. Choosing $|c_1| \geq |c_2| \geq |c_3| \geq \ldots$, we have*

$$T = \sum_{i=1}^{\infty} c_i P_i,$$

with the series converging in norm.

Proof. We have already proved that for $T \in B(H)$ normal and for a and b distinct eigenvalues, we have $ker(T - aI) \perp ker(T - bI)$. Moreover, we know that for T compact, every eigenvalue corresponds to a finite dimensional eigenspace. For $N \in \mathbb{N}$, we set $M := \sum_{i=1}^{N} H_{c_i}$ and notice that the previous theorem implies that M reduces T, but also $\sum_{i=1}^{N} c_i P_i$. We then notice that $(\sum_{i=1}^{N} c_i P_i)|_{M^\perp} = 0$ and that $(T - \sum_{i=1}^{N} c_i P_i)|_M = 0$. Consequently, again by the previous theorem, the fact that $|c_n|$ is decreasing, and part (c) of theorem 3.53, we conclude

$$\begin{aligned}\left\| T - \sum_{i=1}^{N} c_i P_i \right\| &= \max\left\{ \left\| (T - \sum_{i=1}^{N} c_i P_i)|_M \right\|, \left\| (T - \sum_{i=1}^{N} c_i P_i)|_{M^\perp} \right\| \right\} \\ &= \max\{0, \|T|_{M^\perp}\|\} \\ &= \|T|_{M^\perp}\| \\ &= |c_{N+1}|,\end{aligned}$$

Letting $N \to \infty$ completes the proof. □

References

[1] W. Rudin, Functional Analysis, 2nd edition, McGraw Hill, 1991.