# Functional Analysis II, Math 7321
# Lecture Notes from April 11, 2017

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## 4 Test Functions and Distributions

Consider the Hilbert space $H = L^2(\mathbb{R})$ and a bounded linear functional $f$ on $H$, there exists $y_f \in H$ such that $f(x) =< x, y_f >$ for all $x \in H$. We wish to characterize continuous linear functional on other function spaces in a similar way.

### 4.A Test Functions

We denote some notations that we will use through out this chapter.

- $\Omega$ is a nonempty subset of $\mathbb{R}^n$ for a positive integer $n$.
- We use multi index notation $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_0^n$ and $D^\alpha = \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_n^{\alpha_n}}$ with order $|\alpha| = \sum_{j=1}^n \alpha_j$, where $\mathbb{Z}_0$ is the set of non-negative integers.
- We say $f \in C^\infty(\Omega)$ if any derivative $D^\alpha f \in C(\Omega)$ for each $\alpha \in \mathbb{Z}_0^n$.
- $supp(f) = \overline{\{x : f(x) = 0\}}$, the support of $f$.

The following are examples of functions with a compact support.

*4.1 Examples.* (a) A characteristic function on a compact set $K \subseteq \Omega$

$$\chi_K(x) = \begin{cases} 1 & x \in K \\ 0 & \text{otherwise} \end{cases}$$

is a function with compact support. But it is not in $C^\infty(\Omega)$.

(b) For a positive number $a > 0$,

$$f_a(x) = \begin{cases} \exp(-\frac{a^2}{a^2 - x^2}) & x \in (-a, a), \\ 0 & \text{otherwise} \end{cases}$$

is a function in $C^\infty(\mathbb{R})$ with a compact support ([1], Chapter 4, page 169).

(c) Define

$$\phi(x) = \begin{cases} \exp\left(\dfrac{-1}{x^2 - x}\right) & \text{if } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

This function is in $C^\infty(\mathbb{R})$ and $supp(f) = [0, 1]$.

We are interested a subset of $C^\infty(\Omega)$ which contains all functions with a compact support. This set is not empty as we saw from the previous examples $(b)$ and $(c)$. The following defines the family of functions in $C^\infty(\Omega)$ with a compact support.

**4.2 Definition.** Let $\Omega \subseteq \mathbb{R}^n$ be open set. For a compact subset $K \subseteq \Omega$,

$$\mathcal{D}_K = \{f \in C^\infty(\Omega) : supp(f) \subseteq K\}.$$

The space of functions, which are $C^\infty(\Omega)$ and have a compact support, is defined as

$$\mathcal{D}(\Omega) = \{f : f \in \mathcal{D}_K \text{ for some compact set } K \subseteq \Omega\}.$$

This space is called **the space of test functions** and its elements are called **test functions**.

We recall that there is an increasing sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots$, such that $\bigcup_{j=1}^\infty K_j = \Omega$. Thus, $\mathcal{D}(\Omega) = \bigcup_{i=1}^\infty \mathcal{D}_{K_i}$. Last semester, we defined seminorms on $C(\Omega)$ which made $C(\Omega)$ a locally convex topological vector space (see the note on October 13, 2016). Similarly, we introduce seminorms $\{p_N\}_{N=1}^\infty$ on $C^\infty(\Omega)$,

$$p_N(f) = \max\{|D^\alpha f(x)| : x \in \Omega, |\alpha| < N\}.$$

These induces a locally convex topology which renders $C^\infty(\Omega)$ metrizable. In addition, a local base of the topology is given by the sets ([2], page 35)

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < \frac{1}{N}\}.$$

As an example, a continuous linear function on this space, given $x \in \Omega$, then there is $N_0$ such that for each $N \geq N_0$, $x \in K_N$ and by $|f(x)| \leq \max\{|D^\alpha f(x)| : |\alpha| < N\}$, we have that $\wedge_x$ is continuous because if $f_n \to f$ in $C^\infty(\Omega)$, $f_n|_K \to f|K_n$ uniformly as $n \to \infty$.

By this topology, $C^\infty$ and also $\mathcal{D}_{K_i}$ are complete. Unfortunately, in general $\mathcal{D}(\Omega)$ is not complete. Consider $n = 1$, $\Omega = \mathbb{R}$. Let $\phi \in \mathcal{D}(\Omega)$ with $supp(\phi) = [0, 1]$ (an example 4.1.1(c), for instance). Let define $\psi_n(x) = \sum_{j=1}^n \frac{1}{j}\phi(x-j)$. Then $\psi_n(x)$ is a Cauchy sequence but the limit is not in $\mathcal{D}(\Omega)$. The Cauchy property of $\{\psi_n\}|_{n\in\mathbb{N}}$ follows from considering $n, m \in \mathbb{N}, m \geq n$,

$$\psi_m(x) - \psi_n(x) = \sum_{j=n+1}^{m} \psi(x - j).$$

For $N \in \mathbb{N}$,

$$p_N(\psi_m - \psi_n) = \max\{|D^\alpha(\psi_m - \psi_n)(x)| : x \in K_n, |\alpha| \leq N\} = \sum_{j=n+1}^{m} |D^\alpha \frac{1}{j}\psi(x - j)|.$$

By disjoint of support of $D^\alpha(\psi(x_j))$,

$$p_N(\psi_m - \psi_n) \leq \max\{|D^\alpha \frac{1}{j}\psi(x-j)| : j \in \{n+1, ..., n\}\}$$
$$= \frac{1}{n+1}\max\{|D^\alpha \psi(x-j)| : x \in \mathbb{R}, j \in \{n+1, ..., n\}\}.$$

As $m, n \to \infty$, $p_N(\psi_m - \psi_n) \to 0$. However, $\psi_n$ had a limit which its support is $[0, \infty)$ which is not compact.

We want to choose a topology which mimics the metrizable topology locally, but suppresses leakage of the support.

**4.3 Definition.** Let $\Omega$ be a nonempty open subset of $\mathbb{R}^n$.

1. For each compact set $K \subseteq \Omega$, let $\tau_K$ denote the topology of the Frechet space of $\mathcal{D}_K \subseteq \mathcal{D}(\Omega)$.
2. Let $\beta$ be the collection of convex balance set $W \subseteq \mathcal{D}(\Omega)$ such that $\mathcal{D}_K \cap W \in \tau_K$ for each compact set $K \subseteq \Omega$.
3. Let $\tau$ be the union of sets $\phi + W$ with $W \in \beta, \phi \in \mathcal{D}(\Omega)$ i.e.,
$$\tau = \{\phi + W : \phi \in \Omega, W \in \beta\}.$$

We will see $\tau$ is complete topology (but not metrizable).

*4.4 Example.* As an example of a set in $\tau$, consider any $x \in \Omega, c > 0$,

$$W = \{\varphi \in \mathcal{D}(\Omega) : |\varphi(x)| < c\}.$$

Then $W$ is convex and balance. Given $K \subseteq \Omega_0$ compact, then

$$\phi(x)\mathcal{D} \cup W = \begin{cases} \{\varphi \in \mathcal{D}_K : |\varphi(x)| < c\} & \text{if } x \in K, \\ \mathcal{D}_K & \text{otherwise.} \end{cases}$$

So if $x \in K_0$, $\mathcal{D}_K \subseteq \tau_K$; if $x \in K^0$, then $\mathcal{D}_K \cap W = \wedge_x^{-1}((c, -c)) \in \tau_K$ by continuity of $\wedge_x$ on $\tau_K$.

Next, we investigate properties of $\tau$.

**4.5 Theorem.** *1. The collection $\tau$ is a topology and $\beta$ is a local base.*

*2. Equipped with $\tau, \mathcal{D}(\Omega)$ becomes a locally convex topological vector space.*

*Proof.* (1) We have $\phi \in \tau, \mathcal{D}(\Omega) \in \tau$, and by the definition $\tau$ is stable under unions. We only need to check finite intersection. Take $V_1, V_2 \in \tau, phi \in V_1 \cap V_2$. By $W \in \tau$ for any balance convex set as described, we only need to find such a $W$ with $\phi + W \subseteq V_1 \cap V_2$. From $V_1 \in \tau$, we know there are $\phi_i, W_i$ for $i = 1, 2$ such that $\phi_i + W_i \subseteq V_i$ for $i = 1, 2$. Let $K$ be such that $\mathcal{D}(K)$ contains $\phi, \phi_1, \phi_2$. Then by $\mathcal{D}_K \cap W_i$ is open, there is $\sigma_i$ such that $\phi - \phi_i \in (1 - \delta_i)W_i$ for $i = 1, 2$ (Be continued).

□

# References

[1] Boccara, Nino. *Functional analysis: an introduction for physicists*, Academic Press, San Diago, CA, 1990.

[2] Rudin, Walter. *Functional Analysis, 2nd*, McGraw Hill Education, India, 1973.