Last time, we discussed the space of test functions $\mathcal{D}(\Omega)$ on a nonempty open set $\Omega \subset \mathbb{R}^n$, and considered a topology on it which was metrizable but not complete. We then proposed a new topology $\tau$ on $\mathcal{D}(\Omega)$, and will eventually show that $\tau$ is complete but not metrizable. First, we must prove that $\tau$ is in fact a topology.

For convenience, we recall the definition of $\tau$.

4.6 Definition. Let $\Omega \neq \emptyset$ be open in $\mathbb{R}^n$. For each compact $K \subset \Omega$ let $\tau_K$ denote the topology of the Fréchet space $\mathcal{D}_K \subset \mathcal{D}(\Omega)$. Let $\beta$ be the collection of convex, balanced sets $W \subset \mathcal{D}(\Omega)$ such that $\mathcal{D}_K \cap W \in \tau_K$ for every compact $K \subset \Omega$. Define $\tau$ to be the collection of unions of sets of the form $\phi + W$ with $W \in \beta$ and $\phi \in \mathcal{D}(\Omega)$.

4.7 Theorem. The collection $\tau$ is a topology on $\mathcal{D}(\Omega)$ with local base $\beta$. Equipped with $\tau$, $\mathcal{D}(\Omega)$ becomes a locally convex topological vector space.

Proof. Clearly $\emptyset \in \tau$. Also, $\mathcal{D}(\Omega) \cap \mathcal{D}_K = \mathcal{D}_K \in \tau_K$ for all compact $K \subset \Omega$. Because $\mathcal{D}(\Omega)$ is trivially convex and balanced, we see $\mathcal{D}(\Omega) \in \tau$. We also have that $\tau$ is stable under arbitrary unions by definition, so it only remains to show $\tau$ is closed under finite intersections.

Take $V_1, V_2 \in \tau$, and $\phi \in V_1 \cap V_2$. Since $\beta \subset \tau$, if we can find $W \in \beta$ with $\phi + W \subset V_1 \cap V_2$ then we will be done. For $i = 1, 2$, since $V_i \in \tau$, we know there exists some $\phi_i \in \mathcal{D}(\Omega)$ and $W_i \in \beta$ such that $\phi \in \phi_i + W_i \subset V_i$.

Let $K$ be such that $\mathcal{D}_K$ contains $\phi, \phi_1$, and $\phi_2$. By $\mathcal{D}_K \cap W_i$ open in $\tau_K$, there is a $\delta_i > 0$ such that $\phi - \phi_i \in (1 - \delta_i)W_i$ for $i = 1, 2$. So by convexity of $W$,

\[ \phi - \phi_i + \delta_i W_i \subset (1 - \delta_i)W_i + \delta_i W_i = W_i \]

\[ \implies \phi + \delta_i W_i \subset \phi_i + W_i \subset V_i. \]

Letting $W = \delta_1 W_1 \cap \delta_2 W_2$, we see that $W$ is convex, balanced, in $\beta$, and $\phi + W \subset V_1 \cap V_2$. Thus $V_1 \cap V_2 \in \tau$. Therefore $\tau$ is a topology on $\mathcal{D}(\Omega)$.

Moreover, the same argument with $V_1 = V_2$ for any $V_1 \in \tau$ with $0 \in V_1$ shows that there is $W \subset V_1$ with $W \in \beta$. So $\beta$ is a local base.
Next, we must show that $(\mathcal{D}(\Omega), \tau)$ is a topological vector space. By our definitions from 9/20, this entails showing that singletons are closed, the vector space operations are continuous with respect to $\tau$.

Given $\phi \in \mathcal{D}(\Omega)$, let $\phi' \neq \phi$ and define $W = \{ \psi \in \mathcal{D}(\Omega) : p_0(\psi) < p_0(\phi - \phi') \}$. Then $W \in \beta$, and by the triangle inequality we see $\phi \notin \phi' + W$. Thus $\{\phi\}$ is closed.

Lastly, it remains to show the vector space operations are continuous. For this, take $\alpha, \alpha_0 \in \mathbb{K}$ and $\phi, \phi_0 \in \mathcal{D}(\Omega)$. For any $W \in \beta$, there is a $\delta > 0$ such that $\delta \phi_0 \in \frac{1}{2}W$. Choosing $c > 0$ such that $2c(|\alpha_0| + \delta) = 1$ gives that if $|\alpha - \alpha_0| < \delta$ and $\phi - \phi_0 \in cW$, then:

$$(\alpha - \alpha_0)\phi_0 \in \frac{1}{2}W, \quad \alpha(\phi - \phi_0) \in cW \subset (|\alpha_0| + \delta)cW = \frac{1}{2}W$$

$$\implies \alpha\phi - \alpha_0\phi_0 = \alpha(\phi - \phi_0) + \phi(\alpha - \alpha_0) \in \frac{1}{2}W + \frac{1}{2}W = W.$$

By setting $\alpha = \alpha_0 = 1$ we see that vector addition is continuous, and by setting $\phi_0 = \phi$ we see that scalar multiplication is continuous. Thus $(\mathcal{D}(\Omega), \tau)$ is a topological vector space. Moreover, $(\mathcal{D}(\Omega), \tau)$ is locally convex because $\beta$ is a convex local base (see definition of local convexity on 9/20).

\[\square\]

### 4.B Properties of $\mathcal{D}_K$

Before we discuss the properties of the topology $\tau$ on $\mathcal{D}(\Omega)$, we review some properties of $\mathcal{D}_K$, where $K \subset \Omega$ is compact. First, we show a characterization of boundedness.

**4.8 Proposition.** A subset $E \in \mathcal{D}_K$ is bounded iff each seminorm $p_N$ is bounded.

**Proof.** Recall that $\mathcal{D}_K$ may be identified with a subspace of $C^\infty(\Omega)$, where $C^\infty(\Omega)$ is endowed with the metrizable locally convex topology induced by the seminorms $\{p_N\}_{N \in \mathbb{N}}$ discussed last time. This proposition then follows from Theorem 11.5.1 in the notes for 10/11. \[\square\]

Next, we investigate the relationship between the Cauchy property in $\mathcal{D}_K$ and the seminorms that induced the topology.

**4.9 Proposition.** If $(\phi_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}_K$, then for each fixed $N \in \mathbb{N}$, $\lim_{i,j \to \infty} p_N(\phi_i - \phi_j) = 0$.

**Proof.** Recall that the collection $\mathbb{B}$ of all finite intersections of sets of the form $\{ \phi \in C^\infty(\Omega) : p_N(\phi) < \frac{1}{m} \}$ for $N, m \in \mathbb{N}$ is a convex balanced local base for $C^\infty(\Omega)$. Because $p_N \leq p_{N+1}$ for all $N \in \mathbb{N}$, we see that $\mathbb{B} = \{ \phi \in C^\infty(\Omega) : p_N(\phi) < \frac{1}{m} \}$ where $m \in \mathbb{N}$ (no need to take intersections). Since $\mathcal{D}_K$ is a subspace of $C^\infty(\Omega)$, $\mathbb{B} \cap \mathcal{D}_K$ forms a local base for $\mathcal{D}_K$.

Now let $(\phi_i)_{i \in \mathbb{N}}$ be Cauchy in $\mathcal{D}_K$. Fix $N \in \mathbb{N}$, then for any $m \in \mathbb{N}$, by the Cauchy property there exists an $M \in \mathbb{N}$ such that $\phi_i - \phi_j \in \{ \phi \in \mathcal{D}_K : p_N(\phi) < \frac{1}{m} \}$ for all $i, j \geq M$. So we have $p_N(\phi_i - \phi_j) < \frac{1}{m}$ for all $i, j \geq M$. Since $m \in \mathbb{N}$ was arbitrary, we conclude that $\lim_{i,j \to \infty} p_N(\phi_i - \phi_j) = 0$. \[\square\]
The next proposition gives a characterization of convergent sequences in $D_K$. Note that by translation invariance, it is enough to consider sequences which converge to 0.

4.10 Proposition. A sequence $(\phi_i)_{i \in \mathbb{N}}$ in $D_K$ converges to zero iff for each $\alpha \in (\mathbb{Z}_0^+)^n$, $D^\alpha \phi_i \to 0$ in $C(K)$ (i.e., in sup norm).

Proof. First, suppose we have a sequence $(\phi_i)_{i \in \mathbb{N}}$ in $D_K$ that converges to zero. Since $D_K$ is a closed subspace of $C^\infty(\Omega)$, we see that $\phi_i \to 0$ in $C^\infty(\Omega)$ as well. Let $\alpha \in (\mathbb{Z}_0^+)^n$ be a multi-index, and choose $N \in \mathbb{N}$ such that $|\alpha| \leq N$ and $K \subset K_N$. For a fixed $m \in \mathbb{N}$, $V = \{ \phi \in C^\infty(\Omega) : p_N(\phi) < \frac{1}{m} \}$ is an open neighborhood of 0 in $C^\infty(\Omega)$. Since $\phi_i \to 0$, there exists some $M \in \mathbb{N}$ such that $\phi_i \in V$ for all $i \geq M$.

Note that $p_N(\phi) < \frac{1}{m}$ implies that $\max\{|D^\alpha \phi(x)| : x \in K_N, |\alpha| \leq N\} < \frac{1}{m}$, which means that $|D^\alpha \phi(x)| < \frac{1}{m}$ for all $x \in K_N$. Since $K \subset K_N$, we have that for all $i \geq M$, $\sup_{x \in K} |D^\alpha \phi_i(x)| < \frac{1}{m}$. Since $m \in \mathbb{N}$ was arbitrary, we conclude that $\sup_{x \in K} |D^\alpha \phi_i(x)| \to 0$.

Conversely, suppose that $D^\alpha \phi_i \to 0$ in sup norm for all $\alpha \in (\mathbb{Z}_0^+)^n$. Then for any $m \in \mathbb{N}$ and $N \in \mathbb{N}$, there is some $M \in \mathbb{N}$ such that $\sup_{x \in K} |D^\alpha \phi_i(x)| < \frac{1}{m}$ for all $\alpha$ with $|\alpha| \leq N$, and all $i \geq M$. Choosing $N$ such that $K \subset K_N$, we then see that for $i \geq M$, $p_N(\phi_i) < \frac{1}{m}$. By the monotonicity of the seminorms $p_N$, we conclude that $\phi_i \to 0$.

The above properties help to show the next, more important one: that the Heine-Borel property holds in $D_K$.

4.11 Proposition. If $E \subset D_K$ is closed and bounded, then $E$ is compact.

Proof. Suppose $E \subset D_K$ is closed and bounded. Then for each $N \in \mathbb{N}$ there is $M_N \geq 0$ such that $p_N(\phi) \leq M_N$ for each $\phi \in E$ (again by Theorem 11.5.1). Thus $|D^\alpha \phi| \leq M_N$ on $K$ for all $\alpha$ with $|\alpha| \leq N$. Hence if $\beta \in (\mathbb{Z}_0^+)^n$ is such that $|\beta| \leq N - 1$, then $\{D^\beta \phi : \phi \in E\}$ is equicontinuous (by uniform boundedness). So by closedness, boundedness, and equicontinuity we have $\{D^\beta \phi : \phi \in E\}$ is compact for each fixed $\beta$.

This means that, given a sequence $(\phi_i)_{i \in \mathbb{N}}$, we can select successive subsequence so that $(D^\alpha \phi_{i_k})_{k \in \mathbb{N}}$ converges uniformly for each fixed $\alpha$. Since there are only finitely many $\alpha$ with $|\alpha| \leq N$, after passing to only finitely many subsequences we form a subsequence that converges with respect to $\tau_K$. Hence $E$ is sequentially compact, hence compact by metrizability of $D_K$.

To conclude our discussion of $D_K$, we show that it is complete.

4.12 Proposition. $D_K$ is complete.

Proof. Let $(\phi_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in $D_K$. Let $E = \{ \phi_i : i \in \mathbb{N} \}$. Then $\overline{E}$ is closed, and also bounded since $E$ is (by the Cauchy property). So by above, $\overline{E}$ is compact. Since $(\phi_i)_{i \in \mathbb{N}}$ is in $E$, we see that there exists a convergent subsequence $(\phi_{i_k})_{k \in \mathbb{N}}$. Since $(\phi_i)_{i \in \mathbb{N}}$ was Cauchy, it must converge to $\lim_{k \to \infty} \phi_{i_k}$. 

\[\]
4.C Properties of $\mathcal{D}(\Omega, \tau)$

With the above properties of $\mathcal{D}_K$ in mind, we begin to establish properties of the topological vector space $(\mathcal{D}(\Omega), \tau)$.

4.13 Theorem.

(a) A convex balanced subset $V$ of $\mathcal{D}(\Omega)$ is open iff $V \in \beta$.

(b) $\tau_K$ is the trace topology induced by $\tau$ in $\mathcal{D}(\Omega)$.

(c) If $E$ is bounded in $\mathcal{D}(\Omega)$, then $E \subset \mathcal{D}_K$ for some compact $K \subset \Omega$, and for each $N \in \mathbb{N}$ there is $M_N < \infty$ such that for every $\phi \in E$, $p_N(\phi) \leq M_N$.

(d) $\mathcal{D}(\Omega)$ has the property that closed and bounded sets are compact.

Proof. (a): Let $V \in \tau$. Consider $K \subset \Omega$, $K$ compact, and $\phi \in \mathcal{D}_K \cap V$. Then there is a $W \in \beta$ such that $\phi + W \subset V$, hence $\phi + \mathcal{D}_K \cap W \subset \mathcal{D}_K \cap V$. By definition, this means that $\mathcal{D}_K \cap V \in \tau_K$. If in addition $V$ is convex and balanced, then $V \in \beta$ by the definition of $\beta$.

We will continue this proof next time. \qed