Functional Analysis, Math 7321
Lecture Notes from April 13, 2017

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Last time, we discussed the space of test functions $\mathcal{D}(\Omega)$ on a nonempty open set $\Omega \subset \mathbb{R}^n$, and considered a topology on it which was metrizable but not complete. We then proposed a new topology τ on $\mathcal{D}(\Omega)$, and will eventually show that τ is complete but not metrizable. First, we must prove that τ *is* in fact a topology.

For convenience, we recall the definition of τ.

4.6 Definition. Let $\Omega \neq \emptyset$ be open in $\mathbb{R}^n$. For each compact $K \subset \Omega$ let τ_K denote the topology of the Fréchet space $\mathcal{D}_K \subset \mathcal{D}(\Omega)$. Let β be the collection of convex, balanced sets $W \subset \mathcal{D}(\Omega)$ such that $\mathcal{D}_K \cap W \in \tau_K$ for every compact $K \subset \Omega$. Define τ to be the collection of unions of sets of the form $\phi + W$ with $W \in \beta$ and $\phi \in \mathcal{D}(\Omega)$.

4.7 Theorem. *The collection τ is a topology on $\mathcal{D}(\Omega)$ with local base β. Equipped with τ, $\mathcal{D}(\Omega)$ becomes a locally convex topological vector space.*

Proof. Clearly $\emptyset \in \tau$. Also, $\mathcal{D}(\Omega) \cap \mathcal{D}_K = \mathcal{D}_K \in \tau_K$ for all compact $K \subset \Omega$. Because $\mathcal{D}(\Omega)$ is trivially convex and balanced, we see $\mathcal{D}(\Omega) \in \tau$. We also have that τ is stable under arbitrary unions by definition, so it only remains to show τ is closed under finite intersections.

Take $V_1, V_2 \in \tau$, and $\phi \in V_1 \cap V_2$. Since $\beta \subset \tau$, if we can find $W \in \beta$ with $\phi + W \subset V_1 \cap V_2$ then we will be done. For $i = 1, 2$, since $V_i \in \tau$, we know there exists some $\phi_i \in \mathcal{D}(\Omega)$ and $W_i \in \beta$ such that $\phi \in \phi_i + W_i \subset V_i$.

Let K be such that $\mathcal{D}_K$ contains ϕ, ϕ_1,and ϕ_2. By $\mathcal{D}_K \cap W_i$ open in τ_K, there is a $\delta_i > 0$ sucuh that $\phi - \phi_i \in (1 - \delta_i)W_i$ for $i = 1, 2$. So by convexity of W,

$$\begin{aligned} &\phi - \phi_i + \delta_i W_i \subset (1 - \delta_i)W_i + \delta_i W_i = W_i \\ \implies &\phi + \delta_i W_i \subset \phi_i + W_i \subset V_i. \end{aligned}$$

Letting $W = \delta_1 W_1 \cap \delta_2 W_2$, we see that W is convex, balanced, in β, and $\phi + W \subset V_1 \cap V_2$. Thus $V_1 \cap V_2 \in \tau$. Therefore τ is a topology on $\mathcal{D}(\Omega)$.

Moreover, the same argument with $V_1 = V_2$ for any $V_1 \in \tau$ with $0 \in V_1$ shows that there is $W \subset V_1$ with $W \in \beta$. So β is a local base.

Next, we must show that $(\mathcal{D}(\Omega), \tau)$ is a topological vector space. By our definitions from 9/20, this entails showing that singletons are closed, the vector space operations are continuous with respect to τ.

Given $\phi \in \mathcal{D}(\Omega)$, let $\phi' \neq \phi$ and define $W = \{\psi \in \mathcal{D}(\Omega) : p_0(\psi) < p_0(\phi - \phi')\}$. Then $W \in \beta$, and by the triangle inequality we see $\phi \notin \phi' + W$. Thus $\{\phi\}$ is closed.

Lastly, it remains to show the vector space operations are continuous. For this, take $\alpha, \alpha_0 \in \mathbb{K}$ and $\phi, \phi_0 \in \mathcal{D}(\Omega)$. For any $W \in \beta$, there is a $\delta > 0$ such that $\delta\phi_0 \in \frac{1}{2}W$. Choosing $c > 0$ such that $2c(|\alpha_0| + \delta) = 1$ gives that if $|\alpha - \alpha_0| < \delta$ and $\phi - \phi_0 \in cW$, then:

$$(\alpha - \alpha_0)\phi_0 \in \frac{1}{2}W, \qquad \alpha(\phi - \phi_0) \in \alpha c W \subset (|\alpha_0| + \delta)cW = \frac{1}{2}W$$
$$\implies \alpha\phi - \alpha_0\phi_0 = \alpha(\phi - \phi_0) + \phi(\alpha - \alpha_0) \in \frac{1}{2}W + \frac{1}{2}W = W.$$

By setting $\alpha = \alpha_0 = 1$ we see that vector addition is continuous, and by setting $\phi_0 = \phi$ we see that scalar multiplication is continuous. Thus $(\mathcal{D}(\Omega), \tau)$ is a topological vector space. Moreover, $(\mathcal{D}(\Omega), \tau)$ is locally convex because β is a convex local base (see definition of local convexity on 9/20). □

4.B Properties of $\mathcal{D}_K$

Before we discuss the properties of the topology τ on $\mathcal{D}(\Omega)$, we review some properties of $\mathcal{D}_K$, where $K \subset \Omega$ is compact. First, we show a characterization of boundedness.

4.8 Proposition. *A subset $E \in \mathcal{D}_K$ is bounded iff each seminorm p_N is bounded.*

Proof. Recall that $\mathcal{D}_K$ may be identified with a subspace of $C^\infty(\Omega)$, where $C^\infty(\Omega)$ is endowed with the metrizable locally convex topology induced by the seminorms $\{p_N\}_{N\in\mathbb{N}}$ discussed last time. This proposition then follows from Theorem 11.5.1 in the notes for 10/11. □

Next, we investigate the relationship between the Cauchy property in $\mathcal{D}_K$ and the seminorms that induced the topology.

4.9 Proposition. *If $(\phi_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}_K$, then for each fixed $N \in \mathbb{N}$, $\lim_{i,j\to\infty} p_N(\phi_i - \phi_j) = 0$.*

Proof. Recall that the collection $\mathbb{B}$ of all finite intersections of sets of the form $\{\phi \in C^\infty(\Omega) : p_N(\phi) < \frac{1}{m}\}$ for $N, m \in \mathbb{N}$ is a convex balanced local base for $C^\infty(\Omega)$. Because $p_N \leq p_{N+1}$ for all $N \in \mathbb{N}$, we see that $\mathbb{B} = \{\phi \in C^\infty(\Omega) : p_N(\phi) < \frac{1}{m}\}$ where $m \in \mathbb{N}$ (no need to take intersections). Since $\mathcal{D}_K$ is a subspace of $C^\infty(\Omega)$, $\mathbb{B} \cap \mathcal{D}_K$ forms a local base for $\mathcal{D}_K$.

Now let $(\phi_i)_{i\in\mathbb{N}}$ be Cauchy in $\mathcal{D}_K$. Fix $N \in \mathbb{N}$, then for any $m \in \mathbb{N}$, by the Cauchy property there exists an $M \in \mathbb{N}$ such that $\phi_i - \phi_j \in \{\phi \in \mathcal{D}_K : p_N(\phi) < \frac{1}{m}\}$ for all $i, j \geq M$. So we have $p_N(\phi_i - \phi_j) < \frac{1}{m}$ for all $i, j \geq M$. Since $m \in \mathbb{N}$ was arbitrary, we conclude that $\lim_{i,j\to\infty} p_N(\phi_i - \phi_j) = 0$. □

The next proposition gives a characterization of convergent sequences in $\mathcal{D}_K$. Note that by translation invariance, it is enough to consider sequences which converge to 0.

4.10 Proposition. *A sequence $(\phi_i)_{i\in\mathbb{N}}$ in $\mathcal{D}_K$ converges to zero iff for each $\alpha \in (\mathbb{Z}_0^+)^n$, $D^\alpha \phi_i \to 0$ in $C(K)$ (i.e., in sup norm).*

Proof. First, suppose we have a sequence $(\phi_i)_{i\in\mathbb{N}}$ in $\mathcal{D}_K$ that converges to zero. Since $\mathcal{D}_K$ is a closed subspace of $C^\infty(\Omega)$, we see that $\phi_i \to 0$ in $C^\infty(\Omega)$ as well. Let $\alpha \in (\mathbb{Z}_0^+)^n$ be a multi-index, and choose $N \in \mathbb{N}$ such that $|\alpha| \leq N$ and $K \subset K_N$. For a fixed $m \in \mathbb{N}$, $V = \{\phi \in C^\infty(\Omega) : p_N(\phi) < \frac{1}{m}\}$ is an open neighborhood of 0 in $C^\infty(\Omega)$. Since $\phi_i \to 0$, there exists some $M \in \mathbb{N}$ such that $\phi_i \in V$ for all $i \geq M$.

Note that $p_N(\phi) < \frac{1}{m}$ implies that $\max\{|D^\gamma \phi(x)| : x \in K_N,\ |\alpha| \leq N\} < \frac{1}{m}$, which means that $|D^\alpha \phi(x)| < \frac{1}{m}$ for all $x \in K_N$. Since $K \subset K_N$, we have that for all $i \geq M$, $\sup_{x\in K} |D^\alpha \phi_i(x)| < \frac{1}{m}$. Since $m \in \mathbb{N}$ was arbitrary, we conclude that $\|D^\alpha \phi_i\|_{\sup} \to 0$.

Conversely, suppose that $D^\alpha \phi_i \to 0$ in sup norm for all $\alpha \in (\mathbb{Z}_0^+)^n$. Then for any $m \in \mathbb{N}$ and $N \in \mathbb{N}$, there is some $M \in \mathbb{N}$ such that $\|D^\alpha \phi_i\|_{\sup} < \frac{1}{m}$ for all α with $|\alpha| \leq N$, and all $i \geq M$. Choosing N such that $K \subset K_N$, we then see that for $i \geq M$, $p_N(\phi_i) < \frac{1}{m}$. By the monotonicity of the seminorms p_N, we conclude that $\phi_i \to 0$. □

The above properties help to show the next, more important one: that the Heine-Borel property holds in $\mathcal{D}_K$.

4.11 Proposition. *If $E \subset \mathcal{D}_K$ is closed and bounded, then E is compact.*

Proof. Suppose $E \subset \mathcal{D}_K$ is closed and bounded. Then for each $N \in \mathbb{N}$ there is $M_N \geq 0$ such that $p_N(\phi) \leq M_N$ for each $\phi \in E$ (again by Theorem 11.5.1). Thus $|D_\alpha \phi| \leq M_N$ on K for all α with $|\alpha| \leq N$. Hence if $\beta \in (\mathbb{Z}_0^+)^n$ is such that $|\beta| \leq N-1$, then $\{D^\beta \phi : \phi \in E\}$ is equicontinuous (by uniform boundedness). So by closedness, boundedness, and equicontinuity we have $\{D^\beta \phi : \phi \in E\}$ is compact for each fixed β.

This means that, given a sequence $(\phi_i)_{i\in\mathbb{N}}$, we can select successive subsequence so that $(D^\alpha \phi_{i_k})_{k\in\mathbb{N}}$ converges uniformly for each fixed α. Since there are only finitely many α with $|\alpha| \leq N$, after passing to only finitely many subsequences we form a subsequence that converges with respect to τ_K. Hence E is sequentially compact, hence compact by metrizability of $\mathcal{D}_K$. □

To conclude our discussion of $\mathcal{D}_K$, we show that it is complete.

4.12 Proposition. *$\mathcal{D}_K$ is complete.*

Proof. Let $(\phi_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}_K$. Let $E = \{\phi_i : i \in \mathbb{N}\}$. Then $\overline{E}$ is closed, and also bounded since E is (by the Cauchy property). So by above, $\overline{E}$ is compact. Since $(\phi_i)_{i\in\mathbb{N}}$ is in $\overline{E}$, we see that there exists a convergent subsequence $\{\phi_{i_k}\}_{k\in\mathbb{N}}$. Since $(\phi_i)_{i\in\mathbb{N}}$ was Cauchy, it must converge to $\lim_{k\to\infty} \phi_{i_k}$. □

4.C Properties of $(\mathcal{D}(\Omega), \tau)$

With the above properties of $\mathcal{D}_K$ in mind, we begin to establish properties of the topological vector space $(\mathcal{D}(\Omega), \tau)$.

4.13 Theorem.

(a) A convex balanced subset V of $\mathcal{D}(\Omega)$ is open iff $V \in \beta$.

(b) τ_K is the trace topology induced by τ in $\mathcal{D}(\Omega)$.

(c) If E is bounded in $\mathcal{D}(\Omega)$, then $E \subset \mathcal{D}_K$ for some compact $K \subset \Omega$, and for each $N \in \mathbb{N}$ there is $M_N < \infty$ such that for every $\phi \in E$, $p_N(\phi) \leq M_N$.

(d) $\mathcal{D}(\Omega)$ has the property that closed and bounded sets are compact.

Proof. (a): Let $V \in \tau$. Consider $K \subset \Omega$, K compact, and $\phi \in \mathcal{D}_K \cap V$. Then there is a $W \in \beta$ such that $\phi + W \subset V$, hence $\phi + \mathcal{D}_K \cap W \subset \mathcal{D}_K \cap V$. By definition, this means that $\mathcal{D}_K \cap V \in \tau_K$. If in addition V is convex and balanced, then $V \in \beta$ by the definition of β.

We will continue this proof next time. □