1.1 Last week

- "Highlights" from last term:
  - representation theory
  - spectral theory
  - the Gelfand transform

- Connectedness and Banach algebras.

Recall: For a Banach algebra $A$ with unit, $G(A)$ is the group of invertible elements of $A$ and $G_0$ is the connected component of the identity. Then since $G_0$ is a normal subgroup of $G(A)$, we defined the quotient group $\Lambda_A = G(A)/G_0$, and called it the index group of $A$.

1.2 Warm-up

Let $X$ be a compact Hausdorff space, and $A = C(X)$. Then

$$G(A) = \{ f \in A : f(x) \neq 0 \text{ for all } x \in X \}.$$ 

Because $A$ is abelian, we deduce that $G_0 = e_A$. Then if $f \in G_0$, we have $f = e^g$ for some continuous function $g$ on $X$. Moreover we can define a function $g_\lambda(x) = e^{\lambda g(x)}$ that interpolates continuously between $g_0(x) = 1$ and $g_1(x) = f$; thus, $1$ and $f$ are homotopic. Conversely, if $f$ is homotopic to $1$, then $f \in G_0$. This shows that $G_0$ is concretely given by the class of functions homotopic to any constant non-zero function.

Similarly, $f_1 \sim f_2$ ($f_1$ and $f_2$ are in the same connected component) if they are homotopic. Namely, $f_1$ and $f_2$ belonging to the same coset of $\Lambda_A$ is equivalent to $f_1$ and $f_2$ being homotopic.

See 2.16 on page 35 in ([Dou91]) for more on this.

1.3 Examples

Let $T = \{ z \in \mathbb{C} : |z| = 1 \}$, $e_n(z) = z^n$ for $n \in \mathbb{Z}$, $P = \text{Span}(e_n)_{n=0}^\infty$, and $T = \text{Span}(e_n)_{n=-\infty}^{\infty}$. We call $P$ and $T$ the spaces of (analytic) polynomials and trigonometric polynomials, respectively. Then in the case of the trig. polynomials, the Banach algebra obtained from closure in the supremum norm is $A \equiv T^\infty = C(T)$ and the spectrum is $\Gamma_A = T$. 

To see why this is true, note that \( \chi \in \Gamma_A \Rightarrow \chi(e_1) = z \) and \( \chi(e_n) = z^n \), and these stay bounded if and only if \( |z| \leq 1 \). Moreover, \( \chi(e_n) = z^{-n} \) stays bounded if and only if \( |z| \geq 1 \), so it must be that \( |z| = 1 \). Hence, \( \chi(e_1) = z \in \mathbb{T} \), and for \( \tau \in T \), \( \chi(\tau) = \tau(z) \). We conclude that the elements in \( \Gamma_A \) are the “point evaluations”, \( \chi = \delta_z \), and \( T \) is dense in \( C(\mathbb{T}) \) by the Stone-Weierstraß theorem.

Now write \( A = P^\infty \). We (only) get continuous homomorphisms for \( |z| \leq 1 \), and they are point evaluations, \( \chi(p) = p(z) \), as for \( C(T) \). Hence, we can consider \( \Gamma_A \) as point evaluations of analytic polynomials on \( D \equiv \{ z \in \mathbb{C} : |z| \leq 1 \} \).

See 2.50 on page 47 in ([Dou91]) for more on this.

This identification of \( \Gamma_A \) as \( D \) is useful for invertibility:

\[
f \in C(X) \text{ is invertible } \iff f(x) \neq 0 \text{ for all } x \in X
\]

Thus \( f(z) = z \) is NOT invertible in \( P^\infty \) because \( f(0) = 0 \) and \( \delta_0 \) is in the spectrum!

**Takeaway:** If someone gives you a commutative algebra, look at its spectrum to learn about invertibility.

As another brief example, note that the spectrum of the (noncommutative) algebra \( A \) of \( n \times n \) matrices with complex entries is \( \Gamma_A = \emptyset \). I.e., there are no nontrivial continuous homomorphisms on \( A \). Thus, we may not learn anything from looking at the spectrum.

### 1.4 Commutative \( C^* \)-Algebras and the Gelfand Transform

We start with a lemma.

**1.6 Lemma.** Let \( A \) be a \( C^* \)-algebra, and \( a \in A \) Hermitian. Then \( \sigma(a) \subset \mathbb{R} \).

**Proof.** Without loss of generality, suppose \( A \) has a unit; otherwise, we adjoin one. Let \( \alpha + i\beta \in \sigma(a) \) with \( \alpha, \beta \in \mathbb{R} \). We show that \( \beta = 0 \).

For all \( \lambda \in \mathbb{R} \), \( \alpha + i(\beta + \lambda) \in \sigma(a + i\lambda1) \), and by comparing the norm with its spectral radius, we have \( |\alpha + i(\beta + \lambda)| \leq \|a + i\lambda1\| \iff \alpha^2 + (\beta + \lambda)^2 \leq \|a + i\lambda\|^2 \), hence

\[
\alpha^2 + (\beta + \lambda)^2 = \|a + \lambda1\|\|a - \lambda1\| \quad (a \in A \text{ is Hermitian})
= \|(a + \lambda1)(a - \lambda1)\| \quad (A \text{ is a } C^* \text{-alg.})
= \|a^2 + \lambda^21\|
\leq \|a^2\| + \lambda^2.
\]

Thus we must have \( \alpha^2 + \beta^2 + 2\beta\lambda \leq \|a\|^2 \), and for this to hold for each \( \lambda \) requires \( \beta = 0 \).


### 1.5 Preview for Next Class: The Gelfand Representation and Functional Calculus

We write two versions of the Stone-Weierstraß theorem, which are stated differently than they usually appear in the literature. For examples of such generalized versions of Stone-Weierstraß, see page 121 in ([Rud91]) and page 43 in ([Dou91]).
1.7 Theorem. (i) Let $X$ be compact and $\mathcal{A} \subset C(X)$ a closed $*$-subalgebra that separates points. Then $\mathcal{A} = C(X)$, or there is $x_0 \in X$ such that $\mathcal{A} = \{ f \in C(X) : f(x_0) = 0 \}$.

(ii) Let $X$ be locally compact, $\mathcal{A} \subset C_0(X)$ a closed $*$-subalgebra that separates points and has no common root. Then $\mathcal{A}$ is dense in $C_0(X)$.

References
