Last time

- Index group for $C(X)$
- Spectrum and Gelfand map for $C(T)$ vs disc algebra
- Commutative $C^*$-algebras and spectral theory

1.5 Theorem. Let $A$ be a commutative $C^*$-algebra then $\mathcal{G} : A \to C(\Gamma_A)$ is a $C^*$-isomorphism. In particular, $\|\hat{x}\|_\infty = \|x\|$ and $(\hat{x})^* = \hat{x}^\ast$.

Proof. If $a \in A$ is Hermitian, then $\|\hat{a}\|_\infty = r(a) = \|a\|$ (since the spectral is real) by $a$ being normal and properties of the Gelfand map. From the preceding lemma, $\hat{a}(\Gamma_A) \subset \sigma(a)$ and we see that $\hat{a}$ is real valued. We conclude for $x = b + ic$ with $b, c$ Hermitian,

$$(\hat{x}) = \overline{b - ic} = \hat{b} - i\hat{c} = (\hat{b} + i\hat{c})^* = \hat{x}^*$$

This shows that the Gelfand transform is an isometric inventive homomorphism because

$$\|\hat{x}\|_\infty^2 = \|\hat{x} \ast \hat{x}\|_\infty$$
$$= \|\hat{x} \ast \hat{x}\|_\infty \text{ (Hermitian)}$$
$$= \|x \ast x\|_\infty$$
$$= \|x\|^2$$

It remains to show that $\mathcal{G}$ is onto. From $\mathcal{G}$ being an isomerty $\mathcal{G}(A) \subset C(\Gamma_A)$ is a complete subalgebra invariant under conjugation (REASON: any cauchy sequence in $\mathcal{G}(A)$ is also a cauchy sequence in $C(\Gamma_A)$ which is complete. Hence the sequence converges in $C(\Gamma_A)$). But $\mathcal{G}$ being an isomerty, the limit is in $\mathcal{G}(A)$).

By $\Gamma_A \in \mathcal{A}'$, $\mathcal{G}(A)$ separates points. To see this let $\chi \neq \chi'$ be two distinct points in $\Gamma_A$. Then there must exist $a \in A$ such that $\chi(a) \neq \chi'(a)$. Finally, since $0 \notin \Gamma_A$, there is no common root for $\mathcal{G}(A)$. Now using the Stone-Weierstrass theorem gives us that $\mathcal{G}(A) = C(\Gamma_A)$.
We can also use Gelfand’s representation to study commutative C∗-subalgebras of C∗-algebras.

1.6 Theorem. Let α be a normal element of a C∗-algebra with unit and C∗(α) the ∗-algebra generated by 1 and α. Then

\[ C^∗(α) \cong C(σ(α)) \]

where the isomorphism maps α to \( \text{id}_{σ(α)} \).

Proof. From \( C^∗(α) = \overline{\text{span}}\{α^n(α^m)^* : n, m \geq 0\} \), it follows from continuity of multiplocation and \( α^*α = αα^* \) that \( C^∗(α) \) is commutative. By Gelfand’s representation theorem \( C^∗(α) \cong C(Γ_{C^∗(α)}) \).

We have \( \hat{1}(χ) = 1 \) and

\[ \hat{α} : Γ_{C^∗(α)} \rightarrow σ_{C^∗(α)}(α) \]

\[ \hat{α}(χ) = χ(α) \]

We show that \( \hat{α} \) is a homeomorphism (for which we show that continuous maps has continuous inverse).

Since \( C(Γ_{C^∗(α)}) \) separates points of \( Γ_{C^∗(α)} \) and the C∗-algebra is generated by 1 and α, \( \hat{α} \) must already separate points (REASON: if not then \( α, α^*, a^n(α^m)^* \) and hence \( \overline{\text{span}}\{a^n(α^m)^* : n, m \geq 0\} \) will not separate points which is not true because \( C^∗(α) \) does so). Thus \( \hat{α} \) is 1-1.

Now using that \( Γ_{C^∗(α)} \) is compact, \( \hat{α} \) being continuous sends \( Γ_{C^∗(α)} \) to a compact set, hence any closed subset of \( Γ_{C^∗(α)} \) is mapped to closed set. Since complements are preserved hence any open subset of \( Γ_{C^∗(α)} \) is mapped to open set. So we get that \( \hat{α} \) has continuous inverse.

It remains to show that \( σ(α) = σ_{C^∗(α)}(α) \). By the inclusion of C∗-algebras,

\[ σ(α) \subseteq σ_{C^∗(α)}(α) \]

To see the reverse inclusion, we assume \( λ \in σ_{C^∗(α)}(α) \setminus σ(α) \) then there is

\[ b = (α - λ1)^{-1} \in A \]

Let \( m > \|b\| \) and choose \( f \in C(σ_{C^∗(α)}(α)) \) with \( f(λ) = m \) and \( |f(z)(z - λ)| \leq 1 \) for all \( z \in σ_{C^∗(α)}(α) \). To find such a function, we specialize to \( \text{ran}(f) \subset [0, m] \) and let \( f_{|B_{1/m}(λ)} = 0 \).

Now using the inverse of Gelfand’s map, which is given by

\[ G^{-1} : C(σ_{C^∗(α)}(α)) \rightarrow A \]

then we get with \( g(z) = f(z)(z - λ) \).

\[ m = \|f\|_∞ \]

\[ = \|G^{-1}(f)\| \]

\[ = \|G^{-1}(f)(α - λ1)b\| \]

\[ = \|G^{-1}(g)b\| \]

\[ ≤ \|G^{-1}(g)||b|| \]

\[ = \|g\|_∞ ||b|| \]

\[ ≤ ||b|| \]

contradicting our choice of \( m \). Hence such a \( λ \) does not exist. \qed
1.7 Corollary. Let \( \mathcal{A} \) be a \( \mathcal{C}^* \)-algebra with unit, \( \mathcal{B} \) be another \( \mathcal{C}^* \)-algebra with unit and \( a \in \mathcal{B} \) be normal then

\[
\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a)
\]

Proof. By inclusion of \( \mathcal{C}^* \)-algebras,

\[
\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{C}^*}[a](a)
\]

and combined with \( \sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{C}^*}[a](a) \) from the proof of previous theorem, we get that the equality \( \sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a) \) holds.

\( \square \)