Last Time:

- Gelfand Representation for Commutative $C^*$-Algebras.
- Properties of Spectrum.

1 Functional Calculus for Operators

**Corollary 1.** If $\mathcal{H}$ is a (complex) Hilbert space and $A \in B(\mathcal{H})$ be normal. Then there is an isometric embedding $\Phi : C(\sigma(A)) \rightarrow B(\mathcal{H})$, with $\Phi(id_{\sigma(A)}) = A$.

**Proof.** This follows from choosing $\Phi = G^{-1}$ in the preceding theorem applied to the $C^*$-algebra generated by 1 and $A$. \qed

**Remark 2.** This allows us to assign an operator $\Phi(f) \equiv f(A)$ to each continuous function $f$ and by isomorphism property, for $f, g \in C(\sigma(A))$

$$f(A)g(A) = (fg)(A),$$

as well as $(f(A))^* = \overline{f}(A)$.

Moreover, $\sigma(f(A)) = f(\sigma(A))$, because

$$\sigma(f(A)) = \sigma(\Phi(f)),$$

$$\approx \sigma_{C(\sigma(A))}(f),$$

$$= f(\sigma(A)).$$
2. Limitations of Functional Calculus

Note: Here \( f(\sigma(A)) = f(\sigma(A)) \) as \( f \) is continuous and \( \sigma(A) \) is compact.

Warm up: Find a \( C^* \)-algebra \( A \) and \( a \in A \) such that \( \sigma_A(a) = [0,1] \).

Example 3. \( A = C([0,1]) \) and \( a \in A \) is defined by \( a(x) = x \).

- Can we find \( A \subseteq B(\mathcal{H}) \)?

2 Limitations of Functional Calculus

To see what limitations the functional calculus has, we consider an example:

Example 4. Summability Properties: Let Hilbert space \( \mathcal{H} \) has orthonormal basis \( (e_j)_{j \in J} \) and let \( x : J \rightarrow \mathbb{C}, j \rightarrow x_j \) be bounded. We claim

\[
Av = \sum_{j \in J} x_j \langle v, e_j \rangle e_j,
\]

defines a normal operator and \( \sigma(A) = \overline{x(J)} \). We note that, the series converges by summability properties of orthonormal basis and \( \|A\| = \|x\|_{\infty} \). By

\[
A^*v = \sum_{j \in J} x_j \langle v, e_j \rangle e_j,
\]

we see that, \( A \) is normal. From orthonormal property \( Ae_j = x_j e_j \), gives eigen-values/vectors. Hence, \( \overline{x(J)} \subseteq \sigma(A) \).

Conversely, if \( \lambda \notin \overline{x(J)} \), then

\[
(A - \lambda I)v = \sum_{j \in J} (x_j - \lambda) \langle v, e_j \rangle e_j,
\]

and hence,

\[
(A - \lambda I)^{-1} = \sum_{j \in J} (x_j - \lambda)^{-1} \langle v, e_j \rangle e_j,
\]

defines a bounded operator \( (A - \lambda I)^{-1} \). This implies that, \( \lambda \notin \sigma(A) \). So, \( \sigma(A) = \overline{x(J)} \). Moreover, we can prove that if \( f \in C(\sigma(A)) \), then

\[
f(A)v = \sum_{j \in J} f(x_j) \langle v, e_j \rangle e_j.
\]
This can be shown first for polynomials and then by taking limits in $C(\sigma(A))$. We can define a functional calculus beyond the range of Gelfand’s representation theorem in this case.

If $E \subset C$ is closed, then we define

$$P_E v = \sum_{x_j \in E} \langle v, e_j \rangle e_j$$

as the spectral projection associated with $E$. We see that

$$P_E^2 v = P_E \left( \sum_{x_i \in E} \langle v, e_i \rangle e_i \right)$$

$$= \sum_{x_j \in E} \left( \sum_{x_i \in E} \langle v, e_i \rangle e_i, e_j \right) e_j$$

$$= \sum_{x_j \in E} \left( \sum_{x_i \in E} \langle v, e_i \rangle \langle e_i, e_j \rangle \right) e_j$$

$$= \sum_{x_j \in E} \langle v, e_j \rangle e_j \quad \text{(because } (e_j)_{j \in J} \text{ is orthonormal)}$$

$$= P_E$$

Thus $P_E^2 = P_E$. This defines an orthogonal projection. Also, we see that $P_E$ commutes with $A$, which proved as follows:

$$(P_E A)v = P_E (Av)$$

$$= P_E \left( \sum_{j \in J} x_j \langle v, e_j \rangle e_j \right)$$

$$= \sum_{x_i \in E} \left( \sum_{j \in J} x_j \langle v, e_j \rangle \langle e_j, e_i \rangle \right) e_i$$

$$= \sum_{x_i \in E} \sum_{j \in J} x_j \langle v, e_j \rangle \langle e_j, e_i \rangle e_i$$

$$= \sum_{x_i \in E} x_i \langle v, e_i \rangle e_i .$$
and

\[(AP_E)v = A(P_Ev)\]

\[= A \left( \sum_{x_j \in E} \langle v, e_j \rangle e_j \right)\]

\[= \sum_{i \in J} x_i \left( \sum_{x_j \in E} \langle v, e_j \rangle e_j, e_i \right) e_i\]

\[= \sum_{x_j \in E} \sum_{i \in J} x_i \langle v, e_j \rangle \langle e_j, e_i \rangle e_i\]

\[= \sum_{x_j \in E} x_j \langle v, e_j \rangle e_j .\]

Thus, \((P_EA)v = (AP_E)v\). Also, we have

\[\sigma(A|_{P_E(H)}) = E \cap x(J) \subset E .\]

Hence, this operator \(A|_{P_E(H)}\) is the "the piece" of \(A\) for which the spectrum is in \(E\). It would be nice to have \(P_E = f(A)\) for some \(f \in C(\sigma(A))\). Then we would have

\[f(x) = \begin{cases} 1, & x_j \in E \\ 0, & x_j \notin E \end{cases} .\]

So, \(f\) is \([0, 1]\)-valued. Thus, \(\sigma(A) = f^{-1}(\{0\}) \cup f^{-1}(\{1\})\) splits \(\sigma(A)\) in two closed and open subsets. If \(\sigma(A) \cap E\) is not closed and open then, we can not find such a function \(f\) in \(C(\sigma(A))\).

Example: Take \(J = \mathbb{N}\), and \(x_j\) to enumerate \(\mathbb{Q} \cap [0, 1]\). Then we can not find \(P_E\) other than \(P_E = O\) or \(P_E = I\) corresponding to \(f \in C([0, 1])\). Because, if \(P_E \neq O\) then there is \(0 \neq y \in E\) such that \(P_Ev = y\) for some \(v \in \mathcal{H}\). This implies that

\[\sum_{x_j \in E} \langle v, e_j \rangle e_j = y\]

\[\sum_{x_j \in E} \langle v, e_j \rangle \langle e_j, e_i \rangle = \langle y, e_i \rangle, \quad i \in J\]

\[\langle v, e_i \rangle = \langle y, e_i \rangle, \quad i \in J, \quad \text{as } (e_j)_{j \in J} \text{ is orthonormal},\]

\[\implies y = v.\]

So, \(P_Ev = v\) for all \(v\), and this implies \(P_E = I\).

Thus, \(P_E = O\) or \(P_E = I\).