Last time

- Functional calculus and its limitations

**Warm up/Recap:** Topologies on $B(\mathcal{H})$

Definitions:

- **Norm topology:** Topology induced by the supremum norm on $B(\mathcal{H})$.

- **Strong operator topology, denoted SOT:** coarsest topology such that $\forall v \in \mathcal{H}$, the map $T_v : B(\mathcal{H}) \to B(\mathcal{H})$; $A \mapsto Av$ is continuous.

- **Weak operator topology, denoted WOT:** coarsest topology such that $\forall v, w \in \mathcal{H}$, $\lambda_{v,w}(A) = \langle Av, w \rangle$ is continuous.

Characterization using sequences: (Let $A \subset B(\mathcal{H})$)

- **Norm topology:** $X \in \overline{A} \iff$ there exist $\{x_n\}$ sequence in $A$ such that $x_n \to x$

- **Strong operator topology:** We know $X \in \overline{A}^{SOT} \iff$ for any strongly open set $U$ and $X \in U$, we have $U \cap X \neq \emptyset$. Using basis of topology, for each open $U$ and $X \in U$ we can find a finite set of vectors $\{v_1, v_2, \cdots, v_n\}$ in $\mathcal{H}$ and $\epsilon > 0$,

$$V := \{ Y \in B(\mathcal{H}) : \| (Y - X)v_j \| < \epsilon, \forall j \in \{1, 2, \cdots, m\} \}$$

with $X \in V$ and $V \subset U$. Hence,

$X \in \overline{A}^{SOT} \iff \forall\{v_1, v_2, \cdots, v_n\} \subset \mathcal{H}, \epsilon > 0$, we can find $Y \in X$ such that $\| (Y - X)v_j \| < \epsilon > 0$ for all $j \in \{1, 2, \cdots, m\}$.

- **Weak operator topology, denoted WOT:** $X \in \overline{A}^{WOT} \iff$ there exists $X_n$ sequence in $A$ such that $\forall v, w \in \mathcal{H}$, $\langle X_n v, w \rangle \to \langle Av, w \rangle$ in $\mathbb{C}$.

We return to functional calculus and hope to use weaker topologies to get more functions of operators.

**1.47 Definition.** Let $\mathcal{H}$ be a Hilbert space, $E \subset B(\mathcal{H})$, then we define the commutant of $E$

$$E' := \{ A \in B(\mathcal{H}) : AB = BA, \forall B \in E \}$$
We now have a lemma on the properties of the commutant, (which are strikingly similar to properties of a closed space and its perp).

**1.48 Lemma.** For sets $E, F \subset \mathcal{B}(\mathcal{H})$, we have

1. $E \subset F' \iff F \subset E'$
2. $E \subset E''$
3. $E \subset F \implies F' \subset E'$
4. $E' = E''$
5. $E = E'' \iff$ there is $F \subset \mathcal{B}(\mathcal{H})$ such that $E = F'$

**Proof.**

1. If $E \subset F'$ then for $A \in E, B \in F$ $AB = BA$ and thus by symmetry $F \subset E'$. Swapping $E$ and $F$ give the converse.

2. If in part 1. we choose $F = E'$ then $F \subset E'$, i.e., $E' \subset E'$ implies $E \subset F'$, i.e., $E \subset E''$.

3. If $E \subset F$ let $A \in F'$ then $AB = BA$ for all $B \in F$ and for all $C \in E AC = CA$ since $C \in E \subset F$ hence $A \in E'$ and we have $F' \subset E'$.

4. From part 2. $E' \subset (E')''$ and combining parts 2. and 3. we get $E \subset E'' \implies E'' \subset E'$. Hence $E' = E''$.

5. If $E = F'$ then by part 4. $F' = F'' \implies E = E''$. Conversely, if $E = E'' = (E')'$, then setting $F = E'$ gives $E = E'' = (E')' = F'$.

**1.49 Lemma.** Let $E \subset \mathcal{B}(\mathcal{H})$, then

1. $E'$ is a closed subalgebra of $\mathcal{B}(\mathcal{H})$.
2. If $E$ is commutative, then so is $E''$.
3. If $E$ is invariant under taking adjoints, then so is $E'$.

**Proof.**

1. $\forall A \in E, \{A\}' = \{B \in \mathcal{B}(\mathcal{H}) = AB = BA\}$ a is closed by continuity of multiplication, so $E' = \bigcap_{A \in E}\{A\}'$ is closed. $E$ is a subalgebra since if $C, D \in E'$, then $AC = CA$ and $AB = BA \forall A \in E$

   - for any scalar $k A(kB) = k(AB) = kBA$ hence $kB \in E$.
   - $A(B + C) = AB + AC = BA + BC = (B + C)A$ hence $B + C \in E$.
   - $A(BC) = BAC = BCA = (BC)A$ hence $BC \in E$.

2. If $E$ is commutative, then $E \subset E'$; and by the properties $E'' \subset E'$, so by reversing and taking commutant using properties, $E'' \subset E''$. This implies $E''$ is commutative.

3. This follows from taking adjoint of products as follows. We have $A \in E \iff A^* \in E$. If $B \in E', AB = BA$ for all $A \in E$ and taking adjoints $B^*A^* = A^*B^*$ for all $A^* \in E$. Since $E$ is invariant under adjoints we have $B \in E'$ giving us the result.