Last Time:

- $L^\infty(\mu)$ as a $C^*$-algebra.
- $L^\infty(\mu)$ and associated multiplication operation on $L^\infty(\mu)$.

**Warm up:** Let $T \in B(\mathcal{H})$ be normal. Show that $T = S^*S$ for some $S \in B(\mathcal{H})$ if and only if $\sigma(T) \subset [0, \infty)$.

**Proof.** Assume $T = S^*S$ and let $\lambda \in \sigma(T)$.

So, $T - \lambda 1$ is not invertible which means there is a sequence $(v_n)_{n=1}^\infty$ such that $\|v_n\| = 1$ for each $n \in \mathbb{N}$ and $\|(T - \lambda 1)v_n\| \to 0$.

Hence, $\langle (T - \lambda 1)v_n, v_n \rangle \to 0$ and by $T = S^*S$

$$\langle Sv_n, Sv_n \rangle - \lambda \|v_n\|^2 \to 0,$$

$$\|Sv_n\|^2 - \lambda \|v_n\|^2 \to 0.$$  

Thus, $\lambda \geq 0$ because $\|Sv_n\|^2 \geq 0$ and $\|v_n\|^2 \geq 0$.

Conversely, if $\sigma(T) \subset [0, \infty)$. We know there is a $*$-isomorphism between $C^*(T) = A_T = \text{span}\{T^n(T^*)^m : n, m \geq 0\}$ and $C(\sigma(T))$.

Using $\Phi = \mathcal{G}^{-1}$, we get $S = \Phi(f)$ for $f(x) = \sqrt{x}$ such that $S^2 = T$ and $S^* = \Phi(f^*) = \Phi(f) = S_0$ (since $f$ is real valued ). Therefore, $T = S^*S$ where $S \in B(\mathcal{H})$.

Define $\mathcal{M} = \{\mathcal{M}_\phi : \phi \in L^\infty(\mu)\}$.

**Proposition 1.** $\mathcal{M}$ and $L^\infty(\mu)$ are isometrically isomorphic as $C^*$-algebras.
Proof. By properties listed, \( \phi \rightarrow \mathcal{M}_{\phi} \) is a \( * \)-algebra homomorphism which is contractive.

( To show that the map is an isometry, prove the range of the map is closed sub-algebra of \( \mathcal{B}(\mathcal{H}) \). Since \( \mathcal{B}(\mathcal{H}) \) is complete and since the map is an isomorphism. So, they are isometrically the same.)

We show \( \| \phi \|_{\infty} = \| \mathcal{M}_{\phi} \| \).

Take \( \lambda \in \text{ess} - \text{range}(\phi) \), and let \( \psi = \phi - \lambda \). Then, by \( 0 \in \text{ess} - \text{range}(\phi) \), for each \( \varepsilon > 0 \) we have

\[
E = \{ x \in X : |\psi(x)| < \varepsilon \}
\]

has non-zero measure and

\[
\| \mathcal{M}_{\psi} \chi_E \|^{2} = \| \psi \chi_E \|^{2} = \int_{E} |\psi(x)|^{2}d\mu < \varepsilon^{2} \mu(E), \quad \text{as } |\psi(x)| < \varepsilon \text{ on } E
\]

\[
\Rightarrow \| \mathcal{M}_{\psi} \left( \frac{\chi_E}{\mu(E)} \right) \|^{2} < \varepsilon^{2}.
\]

Thus, \( \mathcal{M}_{\psi} \) is not boundedly invertible, hence neither \( \mathcal{M}_{\phi - \lambda} \), and so \( \lambda \in \sigma(\mathcal{M}_{\phi}) \).

We know that the spectral value satisfies \( |\lambda| \leq r(\mathcal{M}_{\phi}) \leq \| \mathcal{M}_{\phi} \| \). Hence \( \| \phi \|_{\infty} \leq \| \mathcal{M}_{\phi} \| \).

The other side inequality can be obtained as follows: we know that the operator \( \mathcal{M}_{\phi} \) is multiplication operator defined by

\[
\mathcal{M}_{\phi}(f) = \phi f, \quad \forall \ f \in L^{2}(\mu).
\]

Then

\[
\| \mathcal{M}_{\phi}(f) \|^{2} = \int |\phi(x)f(x)|^{2}d\mu(x),
\]

\[
\leq \int |\phi(x)|^{2}|f(x)|^{2}d\mu(x),
\]

\[
\leq \| \phi \|_{\infty}^{2} \int |f(x)|^{2}d\mu(x),
\]

\[
= \| \phi \|_{\infty}^{2} \| f \|^{2},
\]

\[
\Rightarrow \| \mathcal{M}_{\phi} \| = \sup_{\| f \| \leq 1} \| \mathcal{M}_{\phi}(f) \| \leq \| \phi \|_{\infty}.
\]

Thus, this implies that \( \| \phi \|_{\infty} = \| \mathcal{M}_{\phi} \| \).

We want to establish \( L^{\infty}(\mu) = (\mathcal{M}_{C(\chi)})'' \). \( \square \)
Definition 2. An abelian algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ is maximal abelian if it not a proper sub-algebra of a larger abelian algebra of operators on $\mathcal{H}$.

Proposition 3. The $C^*$-algebra $\mathcal{M}$ is maximal abelian.

Proof. By $\mu$ being a probability measure, $L^\infty(\mu) \subset L^2(\mu)$.
We show that if $T \in \mathcal{B}(\mathcal{H})$ commutes with $\mathcal{M}$, that is $T \in \mathcal{M}'$, then $T \in \mathcal{M}$.
So, there is $T \in L^\infty(\mu)$ such that $T = \mathcal{M}_\psi$.
If there is such a $\psi$, it must be $\psi = T1$ where $1 \in L^2(\mu)$. We know for any $\phi \in L^\infty(\mu)$
\[ T\phi = TM\phi 1 = M\phi T1 = M_{\phi\psi}, \]
and
\[ \|\psi\phi\|_2 = \|T\phi\|_2 \leq \|T\|\|\phi\|_2. \]
So, $\|\psi\|_\infty \leq \|T\|$
Because if $\alpha > \|T\|$, setting $E = \psi^{-1}((\alpha, \infty))$, then we get $\mu(E) = \|\chi_E\|^2 = 0$. Otherwise,
\[ \|T\chi_E\| = \|\psi\chi_E\|^2 = \int |\psi|^2 \chi_E d\mu \]
\[ \geq \alpha^2 \int \chi_E d\mu \quad \text{as} \quad |\psi|^2 \geq \alpha^2 \]
\[ = \alpha^2 \mu(E) = \alpha^2 \|\chi_E\|^2 \quad \forall \quad \alpha \geq \|T\|. \]
Now taking $\alpha_n \downarrow \|T\|$, e.g. $\alpha_n = \|T\| + \frac{1}{n}$, gives $\|\psi\|_\infty \leq \|T\|$. By span$\{\chi_E : E \text{ is measurable}\}$ dense in $L^2(\mu)$ and $T$ bounded/ continuous, we have $Tf = \psi f$ for each $f \in L^2(\mu)$. Hence, $T = \mathcal{M}_\psi$. \qed

From this we observe a consequence for the spectrum.

Corollary 4. If $\psi \in L^\infty(\mu)$, then $\text{ess - range}(\phi) = \sigma_M(M_\phi) = \sigma(M_\phi)$, where $M_\phi$ is operator in $\mathcal{B}(\mathcal{H})$.

Proof. If $M_\phi$ is as given for some $\phi \in L^\infty(\mu)$ and $\lambda \in \rho(M_\phi)$ i.e. $(M_\phi - \lambda id)^{-1}$ exists, then $M_\phi$ commutes with $(M_\phi - \lambda)^{-1}$. Since by previous proposition, $\mathcal{M}$ is maximal abelian, so $(M_\phi - \lambda)^{-1} \in \mathcal{M}$. Thus, $\sigma_M(M_\phi) = \sigma(M_\phi)$. Together with $\text{ess - range}(\phi) = \sigma_M(M_\phi)$, we get the identity. \qed