Last Time

- Towards extended functional calculus.

Finishing the theorem from last time-

**Warm-up** Let \((K, \rho, \mu)\) be a Borel measure space for a compact set \(K \subset \mathbb{C}\). Let \(\mathcal{H} = L^2(\mu)\), then

\[ M_{C(K)} \equiv \{ M_\phi : \phi \in C(K) \} \]

forms a commutative \(C^*\)-algebra on \(\mathcal{H}\).

**Q1.** Does \(M_{C(K)}\) have a cyclic vector?

The answer is Yes. Take \(f = 1\) (the characteristic function on \(K\)). Then \(M_\phi(f) = M_\phi(1) = \phi\) for any \(\phi \in C(K)\) and by \(C(K)\) being a dense linear subspace in \(L^2(\mu)\), we have a cyclic vector.

Q1. If \(\phi \in C(K)\), find conditions such that for \(T = M_\phi\), \(\Delta_T = \text{span}\{T^n(T^*)^m ; n, m \geq 0\}^\parallel\) has a cyclic vector.

If we can show that \(\text{span}\{\phi^n\bar{\phi}^m\}\) is dense in \(L^2(\mu)\), then we have a cyclic vector given by \(1\) (Consider the case when \(\phi = 1\), then since we know that \(M_{C(K)} = L^2(\mu)\). We also have that \(M_\phi(f) = f\) for any \(f \in L^2(\mu)\). Since \(K\) is compact and using Stone-Weierstrass Theorem, since \(M_\phi \in C(K)\) is a separating subset of \(C(K)\). Then the complex unital \(*\)-algebra generated by \(M_\phi\) namely \(\Delta_T\) is dense in \(C(K)\). Now since \(C(K)\) is dense in \(L^2(\mu)\), we say that \(\Delta_T\) is a dense linear subspace in \(L^2(\mu)\) and has a cyclic vector \(1\).

![Diagram](image)

**2.5 Remark.**

\[ \mathcal{A}_T \xrightarrow{\mathcal{G}} C(\Gamma) = C(\sigma(T)) \]

\[ \{T\}'' \xrightarrow{\mathcal{G}'} L^\infty(\mu) \]

The above diagram commutes where \(\mu\) is a Borel measure of \(\sigma(T)\) and \(\mathcal{G}\) is an isometry in terms of \(C^*\)-algebra and the map from \(\mathcal{A}_T\) to \(\{T\}''\) is an embedding.

**2.6 Theorem.** Let \(T\) be a normal operator in \(\mathcal{H}\), \(\mathcal{A}_T\) has cyclic vector, then there is a positive regular Borel measure \(\nu\) on \(\mathbb{C}\), \(\text{supp}(\nu) = \sigma(T)\), isometric isomorphism \(\gamma\) from \(\mathcal{H}\) to \(L^2(\nu)\) such that \(\mathcal{G}'(A) = \gamma A \gamma^{-1}\) is a \(*\)-isometric isomorphism from \(\mathcal{A}_T''\) to \(L^\infty(\nu)\), and \(\mathcal{G}'\) is a \(*\)-isomorphic isomorphism from \(\mathcal{A}_T''\) to \(L^\infty(\nu)\).
Moreover, if there is a measure $\nu_1$ on $C$, $G_1'$ a $\ast$-isomorphism from $A''_T$ to $L^\infty(\nu_1)$ extending $G$, then $\nu_1 \sim \nu$, and

$$L^\infty(\nu_1) = L^\infty(\nu)$$

and $G_1' = G'$

**Proof.** We had taken $f$ cyclic to $A_T$. Now consider the map $\psi : C(\sigma(T)) \rightarrow C$ defined as

$$\psi(\phi) \equiv \langle \phi(T)f, f \rangle$$

We have seen that there is a regular Borel measure $\nu$ on $C$ such that for all $\phi \in C(\sigma(T))$,

$$\psi(\phi) \equiv \langle \phi(T)f, f \rangle \int \phi \, d\nu$$

and taking $\phi = 1$ gives that $\nu$ is a probability measure.

**Next we show that** $\sigma(T)$ **is the support of** $\nu$.

If it were not, we would find a relatively open set $\emptyset \neq U \subset \sigma(T)(U = A \cap C$, where $A$ is open in $C$) such that $\nu(U) = 0$. Using Urysohn’s Lemma, taking $\phi \in C(\sigma(T))$ such that $\phi \geq 0, \phi(x) = 1$ for some $x \in U$ and $\phi|_{U^c} = 0$ gives a contradiction because then

$$\langle \sqrt{Q(T)}f, \sqrt{Q(T)}f \rangle = \int \phi \, d\nu = 0$$

So $Q(T)$ annihilates $f$ but $h \in A_T$ and $f$ is cyclic, then $f$ is separating (Lemma from Last time). So we would get $\phi(T) = 0$ but $\phi \neq 0$ as $G$ acts between continuous functions with the sup-norm and the bounded operator. Therefore, $\sigma(T) = \text{supp}(\nu)$. We now define

$$\Phi : A_Tf \mapsto C(\sigma(T))$$

by

$$\phi(T)f \mapsto \phi$$

Then $\Phi$ is an isometry if we equip $A_Tf$ with the norm $\|\phi(T)f\| = \|\phi\|_{L^2(\nu)}$. By definition,

$$\|\phi(T)f\|^2 = \langle (\phi(T))^\ast\phi(T)f, f \rangle$$

$$= \langle |\phi(T)|^2f, f \rangle$$

$$= \int_\sigma |(T)|^2 \, d\nu$$

$$= \|\phi\|_{L^2}^2$$

The norm on $A_Tf$ coincides with the norm on $H$. So, we have $\Phi$ is an isometry that extends to a map defined on $H$.

We also note that we can get any continuous functions being dense in $L^2(\nu)$ and included in $\text{Ran}(\Phi)$, the range of the extended isometry is all of $L^\infty(\nu)$, so $\Phi$ is unitary.

Next, we define $G'$ from $T''$ to $B(L^2(\nu))$ by

$$G'(A) = \Phi A \Phi^{-1}$$
Next we show that $G'|_{\mathcal{A}_T} = G$ (i.e., $G'$ extends the Gelfand transform).

For any $\phi \in C(\sigma(T))$, $g \in C(\sigma(T))$,

$$G'(\phi(T))g = (\Phi \phi(T) \Phi^{-1})g$$
$$= \Phi(\phi(T))g(T)f$$
$$= \Phi(\phi g(T))f$$
$$= \phi g$$
$$= M_\phi g$$

i.e., any continuous function of $T$ corresponds to a multiplication operator. It is true for all $g \in C(\sigma(T))$ and then by continuity (and boundedness) of $G'(\phi(T))$, this identity holds for each $g \in L^2(\nu)$. Now using $\mathcal{A}'' = \overline{\mathcal{A}_T}^{\text{WOT}}$ (By Bicommutant theorem) and $\overline{C(\sigma(T))}^{W^*} = L^\infty(\nu)$ (as $\overline{C(\sigma(T))}^{W^*} = \overline{C(\sigma(T))}^{\text{WOT}}$), we get that $G'$ is a $*$-isometric isomorphism between $\mathcal{A}''$ and $L^\infty(\nu)$.