Last time

- Maximality and cyclic vectors.
- Reproducing kernels.

We continue with the proof of the lemma from last time.

2.2 Lemma. Let $\mathcal{M}$ be a set and $\mathcal{H} \subseteq \mathbb{C}^{\mathcal{M}}$ a Hilbert space such that for each $m \in \mathcal{M}$,

$$\Lambda_m : f \mapsto f(m)$$

is a continuous linear functional, then there exists a function $K : \mathcal{M} \times \mathcal{M} \to \mathbb{C}$ with the following properties:

1. For $y \in \mathcal{M}$, $K_y \equiv K(\cdot, y) \in \mathcal{H}$ gives for all $f \in \mathcal{H}$,

$$\langle f, K_y \rangle = f(y).$$

2. For every finite $\{x_1, \cdots, x_n\} \subset \mathcal{M}$, the matrix $(K(x_j, x_l))_{j,l=1}^n$ is positive semidefinite.

3. The set $\{K_y : y \in \mathcal{M}\}$ is total in $\mathcal{H}$, i.e. has dense linear span in $\mathcal{H}$.

4. If $\{e_j\}_{j \in J}$ is an orthonormal basis and $x, y \in \mathcal{M}$, we have that

$$K(x, y) = \sum_{j \in J} e_j(x) \overline{e_j(y)}.$$ 

Proof. We had already shown (1) and (2). The property (3) follows from $f \perp \{K_m : m \in \mathcal{M}\}$ implying $f(m) = 0$ for each $m \in \mathcal{M}$, so the orthogonal complement of $\{K_m : m \in \mathcal{M}\}$ is $\{0\}$, hence the span of the set of kernel functions is dense. Finally, we obtain (4) from the properties of the orthonormal basis,

$$K_y = \sum_{j \in J} \langle K_y, e_j \rangle e_j$$

and by taking the inner product of both sides with $K_x$. 

\qed
This lemma shows that having a Hilbert space of functions for which all point evaluation functionals gives a positive kernel. The converse holds, too, so each positive kernel $K$ determines a Hilbert space with bounded point evaluation functionals.

2.3 Theorem. Let $K$ be a positive kernel on $M$, then we can equip $\mathcal{H}^0 = \text{span}\{K_y : y \in M\}$ with an inner product

$$\langle \sum_j c_j K_{y_j}, \sum_k d_k K_{y_k} \rangle = \sum_{j,k} c_j d_k K(y_k, y_j).$$

For this space, we have $f(y) = \langle f, K_y \rangle$ for each $f \in \mathcal{H}^0$ and $y \in M$. The completion of $\mathcal{H}^0$ is a Hilbert space of functions on $M$ with continuous point evaluation functionals and $K$ is the corresponding kernel according to the lemma.

Proof. We first need to show that the inner product is well defined. To this end, we note that for $f = \sum_j c_j K_{y_j}$ and $h = \sum_k d_k K_{y_k}$, the value

$$\sum_{j,k} c_j d_k K(y_k, y_j) = \langle f, h \rangle = \sum_{j,k} c_j d_k (y_k(y_j)) = \sum_k d_k f(y_k)$$

depends only on $f$, not on the particular way $f$ was written as a linear combination of kernel functions. Similarly, the above value depends only on $h$, not on the explicit expression in terms of the kernel functions. This permits us to write

$$\langle f, h \rangle = \sum_{j,k} c_j d_k K(y_k, y_j).$$

By the positivity of the kernel, this defines a positive semidefinite sesquilinear form on $\mathcal{H}^0$. From the definition, setting $h = K_y$, we get $\langle f, K_y \rangle = f(y)$. If $\langle f, f \rangle = 0$, then for each $y \in M$, $|f(y)|^2 \leq K(y, y) \langle f, f \rangle = 0$, so $f$ is the zero function. This shows that the sesquilinear form is positive definite.

Let $\mathcal{H}_K$ be the completion of this inner product space, then the set $\{K_y : y \in M\}$ is total, because its span is $\mathcal{H}^0$. Thus, we can identify each $f \in \mathcal{H}_K$ with a function $f : M \to \mathbb{C}$, $f(y) = \langle f, K_y \rangle$, and the point evaluation functionals are obtained as inner products with the kernel functions.

Next, we investigate how the choice of the kernel reflects in properties of the associated Hilbert space.

2.4 Lemma. Let $K$ and $L$ be positive kernels on $M$, then $K + L$ is a positive kernel and

$$\mathcal{H}_{K+L} = \mathcal{H}_K + \mathcal{H}_L \equiv \{f + h : f \in \mathcal{H}_K, h \in \mathcal{H}_L\}.$$ 

Proof. Since the sum of two positive semidefinite matrices is positive semidefinite, $Q = K + L$ is a positive kernel on $M$. Consider $X = \mathcal{H}_K + \mathcal{H}_L$ and define

$$\Phi : X \to \mathcal{H}_K + \mathcal{H}_L, (f, h) \mapsto f + h.$$
The kernel of $\Phi$ consists of $(f, h) \in X : f(y) = h(y) = 0$, which can be rewritten as
$$\ker \Phi = \{(f, g) : ((f, h), (K_y, L_y)) = 0 : y \in M\}.$$ In particular, the kernel is closed and its perp is given by the span of $\{(K_y, L_y) : y \in M\}$. From the decomposition of the direct sum into $\ker \Phi$ and the orthogonal complement, $\ker \Phi^\perp$ is realized as a reproducing kernel on $M$, whose reproducing kernel is given by
$$(x, y) \mapsto \langle (K_y, L_y), (K_x, L_x) \rangle = \langle K_y, K_x \rangle + \langle L_y, L_x \rangle = K(x, y) + L(x, y).$$ This shows $\mathcal{H}_K + \mathcal{H}_L = \mathcal{H}_{K+L}$. \hfill \Box