Last time

- Positivity
- States

Warm up: Let $A = M_n(\mathbb{C})$, $X \in A$, $X = X^*$, then for $\lambda \in \sigma(x)$ there exists $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.

We know that $\lambda \in \sigma(x)$ in $M_n(\mathbb{C})$ means that $\lambda$ is an eigenvalue and there exists $v \in \mathbb{C}^n$ such that $\|v\| = 1$ and $Xv = \lambda v$. Now define $\varphi : A \rightarrow \mathbb{C}$; $A \mapsto \langle Ax, v \rangle$. We have seen that $\varphi \in S(A)$ since $\varphi(1) = 1$, $\|\varphi\| = 1$; and $\varphi(X) = \langle Xv, v \rangle = \lambda$.

1.47 Theorem. Let $A$ be a $C^*$-algebra with unit. Let $x \in A$, $x = x^*$.

- If $\varphi \in S(A)$, then $\varphi(x) \in \mathbb{R}$ and for each $\lambda \in \sigma(x)$, there exists $\varphi \in S(A)$ such that $\varphi(x) = \lambda$.

- An element $x = x^*$ is positive if, and only if $\varphi(x) \geq 0$ for each $\varphi \in S(A)$.

Proof. Let $x = x_+ - x_-$ where $x_+ \geq 0$ and $x_+ x_- = x_- x_+ = 0$. By the lemma characterising states, we have $\varphi(x_+ \geq 0)$, so $\varphi(x) = \varphi(x_+) - \varphi(x_-) \Rightarrow \varphi(x) \in \mathbb{R}$. Next, consider $\lambda \in \sigma(x)$. Let $A_{\delta}$ be the abelian $C^*$-algebra generated by $\{1, x\}$, then $\sigma_{A_{\delta}}(x) = \sigma_{A}(x)$. Using Gelfand for the $A_{\delta}$ $\cong C(\sigma(x))$, we have an isometric $*$-isomorphism $G : A_{\delta} \rightarrow C(\sigma(x))$. For $\lambda \in \sigma(x)$, we have $\delta_{\lambda} : C(\sigma(x)) \rightarrow \mathbb{C}$, $f \mapsto f(\lambda)$ is a state on $C(\sigma(x))$, since $\delta_{\lambda}(1) = 1 = \|\delta_{\lambda}\|$ (1 is the constant function 1), and by isomorphism we get $\nu_{\lambda} = \delta_{\lambda} \circ G$. $\nu_{\lambda}(1) = \delta_{\lambda}(1) = 1$ and by isometry $\|\nu_{\lambda}\| = \|\delta_{\lambda}\| = 1$ hence $\nu_{\lambda}$ is a state on $A_{\delta}$. $\nu_{\lambda}$ can be extended to a state $\nu$ on $A$ by Hahn–Banach such that the extension satisfies $\|\nu\| = \nu(1) = 1$ since $1 \in A_{\delta}$. Thus, $\nu(x) = \nu_{\lambda}(x) = \delta_{\lambda}(1) = \lambda$ giving us the result.

- If $x = x^*$, then we already know that for $\varphi \in S(A)$, $\varphi(x) \geq 0$. Conversely, if $x - x^* \in A$ such that for any $\varphi \in S(A)$, $\varphi(x) \geq 0$, then if $\lambda \in \sigma(x)$, $\lambda \geq 0$ there exists $\varphi$ by the first part such that $\varphi(x) = \lambda \geq 0$ and so $\sigma(x) \subset [0, \infty)$, hence $x \geq 0$.

1.48 Definition. We give a few definitions relating cones and positive elements in a real vector space.
1. (Convex cone) A subset $C$ of a real vector space $V$ is called a convex cone if it is convex and $\mathbb{R}^+ C \subseteq C$.

2. (Dual cone) If $V$ is a topological vector space (such that vector space operations are continuous) and $V'$ is the dual of $V$, then for a cone $C$ in $V$, define the dual cone to $C$ as

$$C' := \{ \alpha \in V' : \alpha(C) \subseteq \mathbb{R}^+ \}.$$ 

3. (Predual cone) If $W \subseteq V'$ is a subset, then

$$(W)' := \{ v \in V : (\forall w \in W) \ w(v) \geq 0 \}$$

is called the predual cone of $W$.

1.49 Remark. If $V$ is a real topological space, $W \subseteq V'$, then $W'$ is closed since $W' = \bigcap_{w \in W} w^{-1}[0, \infty)$ is the intersection of closed sets, where each $w^{-1}[0, \infty)$ is closed since it is a preimage of a closed set under a continuous map.

1.50 Theorem. If $V$ is a real topological vector space and $C \subseteq V$ a closed convex cone, then

$$C = (C')' \text{ and } (C')^\perp = C \cap (-C).$$

Proof. By definition of $C'$, $C \subseteq (C')'$. To show the reverse inclusion, let $x \notin C$. We need to show that $x \notin (C')'$. Using separation properties and geometric Hahn-Banach, there exists a linear functional $\alpha \in V'$ with $\inf_{c \in C} \alpha(c) > \alpha(x)$. From scaling properties of $C$ and linearity of $\alpha$, $\inf_{c \in C} \alpha(c) = 0$, $\alpha(x) < 0$, so $x \notin (C')'$.

We then also get

$$C \cap (-C) = (C')' \cap -(C')' = \{ \alpha \in C : \alpha(a) = 0 \forall a \in C' \} = (C')^\perp.$$

1.51 Lemma. Let $\mathcal{A}$ be a $C^*$-algebra with unit, then $\mathcal{A}^+$ forms a closed convex cone in $\mathcal{A}$.

Proof. Recall $\mathcal{A}^+ = \{ \alpha \in \mathcal{A} : (\forall \varphi \in \mathcal{S}(\mathcal{A}) \varphi(\alpha) \geq 0 \}$, so $\mathcal{A}^+ = S(\mathcal{A})'$.

1.52 Remark. Recall the two theorems used in the proofs above.

1. Hahn-Banach theorem: (Corollary 6.5 from John B. Conway - A Course in Functional Analysis) If $X$ is a normed space over $\mathbb{C}$, $\mathcal{M}$ is a linear manifold in $X$, and $f : \mathcal{M} \rightarrow \mathbb{C}$ is a bounded linear functional, then there is an $F \in X'$ such that $F|_{\mathcal{M}} = f$ and $\|F\| = \|f\|$.

We apply this theorem with $\mathcal{M} = A_{x}$ a subspace of $\mathcal{A}$.

2. (Theorem 2.4.7. from Gert K. Pedersen - Analysis Now) Separation properties and geometric Hahn-Banach: Let $A$ and $B$ be disjoint, nonempty, convex subsets of a topological vector space $X$. If $A$ is open, there is a $\alpha \in X'$ and a $t \in \mathbb{R}$ such that $\text{Re}\alpha(x) < t < \text{Re}\alpha(y)$, for every $x \in A$ and $y \in B$.

We apply theorem with $C^c = A$ which is open since $C$ is closed and $B = \{x\}$. 

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2 The Gelfand–Naimark–Segal (GNS) construction

We start with an observation. Let $\mathcal{A}$ be a $C^*$-algebra with unit, $(\pi, \mathcal{H})$ a representation of $\mathcal{A}$, then we recall $\pi$ is a contraction. For each $v \in \mathcal{H}$, $\|v\| = 1$, we get $\varphi_v(a) = \langle \pi(a)v, v \rangle$ a state, because $\varphi_v(1) = \langle \pi(1)v, v \rangle = \|v\|^2 = 1$, and $\varphi_v(a^*a) = \|\pi(a)v\|^2 \geq 0$. The next goal is to find, for any $\mathcal{A}$ and $\varphi$, a representation $(\pi, \mathcal{K})$ such that there exists $v \in \mathcal{H}$ and $\varphi(a) = \langle \pi(a)v, v \rangle$. 