Last time: From the GNS construction to an isometric isomorphism of $C^*$-algebras.

0 Warm-up

Let $\mathcal{A} = M_n(\mathbb{C})$ and $\varphi \in \mathcal{S}(\mathcal{A})$. Recall that states on a $C^*$-algebra $\mathcal{A}$ are positive, bounded, linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ with unit norm. (We proved this on 2023-03-28.)

0.1 Exercise. Part 1: Show that $\varphi$ is given by

$$\varphi : \mathcal{A} \to \mathbb{C}, \quad \varphi : \mathcal{A} \mapsto \varphi(A) := \text{tr}[AW],$$

where $W \in \mathcal{A}$ is such that $W \geq 0$ and $\text{tr}[W] = 1$.

Part 2: Moreover, $\varphi$ is pure if and only if $W^2 = W$.

Proof of Part 1. First note that since $\mathcal{A}$ is finite-dimensional, we can equip it with the Hilbert-Schmidt norm:

$$\|A\|_{\text{HS}} = \left(\text{tr}[A^*A]\right)^{\frac{1}{2}},$$

as all norms $\|\|_\varphi$ we consider are equivalent. Then, as $\varphi$ is a bounded linear functional on a Hilbert space, we have:

$$\varphi(A) = \text{tr}[AW^*] \quad \text{with some } W \in \mathcal{A},$$

by the Riesz representation theorem for Hilbert spaces. If $I$ denotes the $n \times n$ identity matrix then

$$\varphi(I) = 1$$
$$= \text{tr}[IW^*] = \text{tr}[W^*],$$

since $\varphi$ is a state. Next, we will show that $W \geq 0$. This will imply that all eigenvalues are non-negative real numbers, and $\text{tr}[W] = 1$ follows.

To this end, note that if $X = xx^*$ is a rank-one matrix for some for some $x \in \mathbb{C}^n$, then

$$\varphi(X) = \text{tr}[XW^*]$$
$$= \text{tr}[W^*X] = \langle W^*x, x \rangle.$$
I.e. this is the quadratic form of $W^*$. Hence, if $\lambda \in \sigma(W) \setminus \{0\}$ and $v \in \mathbb{C}^n \setminus \{0\}$ are an eigenvalue-eigenvector pair for $W$, i.e. $Wv = \lambda v$, then
\[
\varphi(vv^*) = \lambda \|v\|^2 \geq 0 \quad \text{by positivity of } \varphi.
\]
Thus $\lambda \in \mathbb{R}^+$, and therefore $W \geq 0$ and $\text{tr}[W] = 1$.

**Proof of Part 2.** Assuming Part 1, and that $W$ also satisfies $W^2 = W$, we can conclude that $W$ is an orthogonal rank-one projection. This means that there exists a vector $v \in \mathbb{C}^n$ with $\|v\| = 1$ such that $W = vv^*$.

Moreover, this is equivalent to the conditions that $\text{tr}[W] = 1$, $W \geq 0$, and $W^2 = W$. To see this, write
\[
v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \implies W := vv^* = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} \overline{v}_1 & \cdots & \overline{v}_n \end{bmatrix}
\]
so that
\[
\text{tr}[W] = \sum_{j=1}^n |v_j|^2 = v^*v = \|v\|^2 = 1, \quad W^2 = (vv^*)vv^* = (v^*v)vv^* = W,
\]
and
\[
\sigma(W) \subset [0, 1) \implies W \geq 0.
\]

**Proof strategy:** We show that if $W$ cannot be written in this form for any $v$, then this is equivalent to $\varphi$ not being a pure state.

So suppose $W$ is not of this form. Then
\[
W = \sum_{j=1}^n \lambda_j P_j \quad \text{by Spectral Theorem},
\]
and $\lambda_j \in [0, 1)$. By $\text{tr}[W] = \sum_{j=1}^n \lambda_j = 1$, we know there exist $\lambda_k, \lambda_\ell \in (0, 1)$ such that we can choose $\varepsilon > 0$ and $\lambda_k \pm \varepsilon$, $\lambda_\ell \mp \varepsilon \in (0, 1)$. But then we can write
\[
\begin{cases}
W' = \sum_{j=1}^n \lambda_j P_j + (\lambda_j + \varepsilon)P_j + (\lambda_j - \varepsilon)P_j \\
W'' = \sum_{\substack{j=1 \\text{or } j \neq k \neq \ell}}^n \lambda_j P_j + (\lambda_j - \varepsilon)P_j + (\lambda_j + \varepsilon)P_j
\end{cases}
\]
Thus $W$ does not represent a pure state.

Conversely, if $W$ does not represent a pure state, we can use the Spectral Theorem to obtain
\[
W = \sum_{j=1}^n \lambda_j P_j
\]
with each $P_j$ an orthogonal rank-one projection, and $P_j P_k = 0 = P_k P_j$ if $j \neq k$.  

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We can then note that
\[ W^2 = \sum_{j=1}^{n} \lambda_j^2 P_j \]
and since
\[
\begin{cases}
\lambda_k^2 < \lambda_k & \text{if } k \in \{1, \ldots, n\} \text{ with } \lambda_k < 1 \\
\lambda_j^2 \leq \lambda_j & \text{for all } j,
\end{cases}
\]
we get
\[
\text{tr}[W^2] = \sum_{j=1}^{n} \lambda_j^2 < \sum_{j=1}^{n} \lambda_j = \text{tr}[W] \implies W^2 \neq W.
\]
Therefore, \( W \) is not an orthogonal rank-one projection.

\[ \square \]

## 1 Representations of pure states are irreducible

We will now continue proving the theorem from the previous class.

**1.1 Theorem.** Let \( \mathcal{A} \) be a \( C^* \)-algebra with unit, and \( \varphi \in S(\mathcal{A}) \). Then \((\pi_\varphi, \mathcal{H}_\varphi)\) (given by the GNS construction) is irreducible if and only if \( \varphi \) is a pure state.

**Proof.** We had shown if \( \varphi \) is not irreducible, then \( \varphi \) is not pure.

Next, we assume \((\pi_\varphi, \mathcal{H}_\varphi)\) is irreducible and let
\[
\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2, \quad \varphi_1 \neq \varphi_2.
\]
By the GNS construction and relationship between
\[
K_1^{(1)} := \varphi_1, \quad K_1^{(2)} := \varphi_2,
\]
we have that
\[
\mathcal{H}_{\varphi_1} \subset \mathcal{H}_{\varphi},
\]
and \( i : \mathcal{H}_{\varphi_1} \to \mathcal{H}_{\varphi} \) intertwines actions of \( \mathcal{A} \) on \( \mathcal{H}_{\varphi} \) and \( \mathcal{H}_{\varphi_1} \).

We observed \( ii^* \pi_\varphi(s) = \pi_\varphi(s) ii^* \). By irreducibility and Schur’s lemma (proved on 2023-02-09),
\[
ii^* \in (\pi_\varphi(\mathcal{A}))' = \mathbb{C} 1.
\]
From this, we know that \( ii^* = \lambda 1, \lambda \in \mathbb{C} \). Then,
\[
ii^* \varphi = i \varphi_1 = \varphi_1
\]
and together with \( 1 = \varphi_1(1) = \lambda \varphi(1) = \lambda \), this shows that \( \lambda = 1 \), and hence \( \varphi_1 = \varphi \). Similarly, one can show that \( \varphi_2 = \varphi \). Thus, \( \varphi \) is an extreme point in \( S(\mathcal{A}) \), and denotes a pure state. \[ \square \]
2 Preview for next class

We will prove in the next class that if $\mathcal{A}$ is an abelian $C^*$-algebra, then the states on $\mathcal{A}$ are precisely the characters on $\mathcal{A}$, i.e., the bounded linear homomorphisms on $\mathcal{A}$. 