Math.1330 – Section 6.1
Sum and Difference Formulas

In this section, we will learn some formulas about finding the trig values for the sum and difference of angles.

For example, we know that

$$\sin(30°) = \frac{1}{2} \quad and \quad \sin(45°) = \frac{\sqrt{2}}{2}$$

The question rises, how can we evaluate

$$\sin(75°) = \sin(30° + 45°) = ?$$

75° is not one of the angles we covered on the unit circle.

Can we compute its trig functions using known angles from the unit circle?

Note:

$$sin(A + B) \neq \sin(A) + \sin(B)$$

$$cos(A + B) \neq \cos(A) + \cos(B)$$

To see this:

$$\sin(30° + 60°) = \sin(90°) = 1$$

$$\sin(30°) + \sin(60°) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}$$

$$\sin(30° + 60°) \neq \sin(30°) + \sin(60°)$$

Sum and Difference Formulas for Sine, Cosine and Tangent

$\sin(A + B) = \sin A \cos B + \sin B \cos A$

$\sin(A - B) = \sin A \cos B - \sin B \cos A$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$tan(A + B) = \frac{tan\ A + tan\ B}{1 - tan\ A\ tan\ B}$

$tan(A - B) = \frac{tan\ A - tan\ B}{1 + tan\ A\ tan\ B}$

Example 1: Given that $\tan(x) = 5$, evaluate $\tan\left(x + \frac{3\pi}{4}\right)$.

Example 2: Given that $\tan(x) = 2$ and $\tan(y) = 6$, evaluate $\tan(x - y)$.

Example 3: Find the exact value of each of the followings using the sum/difference formulas for sine, cosine and tangent:

a. $\sin(75°)$

b. $\sin(15°)$

b. $\cos\left(\frac{7\pi}{12}\right)$

c. $\tan\left(\frac{5\pi}{12}\right)$

Example 4: Suppose that $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$ where $0 < \alpha < \beta < \frac{\pi}{2}$.
Find each of these:

a. $\sin(\alpha + \beta)$

b. $\cos(\alpha - \beta)$

Example 5: Suppose $\cos\alpha = \frac{1}{5}$ and $\tan\beta = -\frac{7}{6}$ where $\pi < \alpha, \beta < 2\pi$. Find

a. $\cos(\alpha + \beta)$

b. $\tan(\alpha + \beta)$

We have used the following formulas/facts in the previous sections:

$$\sin(x+\pi) = -\sin(x)$$

$$\cos(x+\pi) = -\cos(x)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

Now, **we can prove** these formulas by using the sum/difference formulas:

$$\sin(x+\pi) = \sin(x)\cos(\pi) + \sin(\pi)\cos(x) = \sin(x)\,(-1) + 0 \cdot \cos(x) = -\sin(x)$$

$$\cos(x+\pi) = \cos(x)\cos(\pi) - \sin(x)\sin(\pi) = \cos(x)\,(-1) - \sin(x) \cdot 0 = -\cos(x)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2}\right)\cos(x) - \sin(x)\cos\left(\frac{\pi}{2}\right) = 1 \cdot \cos(x) - \sin(x) \cdot 0 = \cos(x)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2}\right)\cos(x) + \sin\left(\frac{\pi}{2}\right)\sin(x) = 0 \cdot \cos(x) + 1 \cdot \sin(x) = \sin(x)$$

Example 6: Simplify each of the following using the sum and difference formulas for sine, cosine and tangent:

a. $\sin(2\theta)\cos(13\theta) + \sin(13\theta)\cos(2\theta)$

b. $\sin 10° \cos 55° - \sin 55° \cos 10°$

c. $\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\sin\left(\frac{7\pi}{12}\right)$

d. $\frac{\tan 40° + \tan 5°}{1 - \tan 40° \tan 5°}$

e. $\frac{\tan 80° - \tan 35°}{1 + \tan 80° \tan 35°}$

f. $\cos(x + 60°) =$

Example 7. Let $f(x) = \sin\left(x + \frac{\pi}{4}\right) - \sin\left(x - \frac{\pi}{4}\right)$. Simplify the formula.
Find the maximum and the minimum values of this function.
State the x-intercepts of this function over the interval $[0, 2\pi)$.

Example 8. In the figure below, $\sin(\widehat{BAD}) = \frac{1}{5}$ and $\sin(\widehat{ABD}) = \frac{3}{5}$.

Find $\sin(\widehat{BDC})$.