Math.1330 – Section 6.2
Double and Half Angle Formulas

We know trigonometric values of many angles on the unit circle.
Can we use them to find values for more angles?

For example, we know all trigonometric values of $45°$;
can we use that information to find trigonometric values of $22.5°$?

Or, if we know that $\sin(x) = \frac{1}{3}$, is there a way to find $\sin(2x)$?

Yes, there is a way evaluating half/double of the angles we know.

Before introducing such formulas that allows us to evaluate different angles, let's emphasize the followings:

$$\sin(2x) \neq 2\sin(x)$$

$$\cos(2x) \neq 2\cos(x)$$

$$\tan(2x) \neq 2\tan(x)$$

To see this, we have:

$$\sin(30°) = \frac{1}{2} \ \rightarrow 2 \cdot \sin(30°) = 2 \cdot \frac{1}{2} = 1$$

$$\sin(2 \cdot 30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$

$$\sin(2 \cdot 30°) \neq 2 \cdot \sin(30°)$$

Double Angle Formulas for Sine, Cosine and Tangent.

Sine Formula:

$$\begin{aligned}\sin(2A) &= \sin(A + A)\\ &= \sin(A)\cos(A) + \sin(A)\cos(A)\\ &= 2\sin(A)\cos(A)\end{aligned}$$

Cosine Formula:

$$\begin{aligned}\cos(2A) &= \cos(A + A)\\ &= \cos(\mathrm{A})\cos(A) - \sin(A)\sin(A)\\ &= cos^2(A) - sin^2(A)\end{aligned}$$

$$\begin{aligned}\cos(2A) &= cos^2(A) - sin^2(A)\\ &= 1 - 2sin^2(A)\\ &= 2cos^2(A) - 1\end{aligned}$$

Tangent Formula:

$$\begin{aligned}\tan(2A) &= \tan(A + A)\\ &= \frac{tan(A)+\tan(A)}{1-\tan(A)\tan(A)}\\ &= \frac{2\tan(A)}{1-\tan^2(A)}\end{aligned}$$

Double-Angle Formulas:

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\begin{aligned}\cos(2x) &= cos^2(x) - sin^2(x)\\ &= 2cos^2(x) - 1\\ &= 1 - 2sin^2(x)\end{aligned}$$

$$\tan(2x) = \frac{2\tan(x)}{1 - tan^2(x)}$$

Example 1. Suppose x is an acute angle such that $\sin(x) = \frac{1}{5}$. Evaluate:

a) $\sin(2x)$

b) $\cos(2x)$

c) $\tan(2x)$

Example 2. Given $\tan(x) = 4$, find $\tan(2x)$.

Example 3. Given $\cos(x) = \frac{1}{5}$, find $\cos(2x)$.

Example 4. Given $\sin(A) = \frac{1}{6}$, find $\cos(2A)$.

Example 5. Given $\tan(x) = 2$ and $0 < x < \frac{\pi}{2}$, find $\sin(2x)$.

Example 6. Given $\cos(\theta) = -\frac{2}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate:

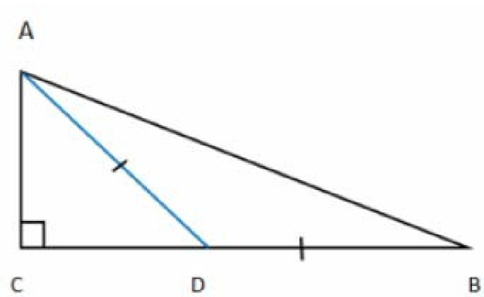
a) $\cos(2\theta)$

b) $\sin(2\theta)$

c) $\tan(2\theta)$

Example 7. In the triangle ABC with the right angle C, $\sin(B) = \frac{1}{4}$ and

|AD|=|BD|. Find $\sin(\widehat{ADC})$.



Example 8: Simplify each:

a. $2\sin(75°)\cos(75°)$

b. $\cos^2\frac{\pi}{9}-\sin^2\frac{\pi}{9}$

c. $\frac{2\tan 15°}{1-\tan^2 15°}$

d. $1-2\sin^2(6A)$

e. $\frac{10\sin(x)\cos(x)}{cos^2(x)-sin^2(x)}$

f. $\frac{1-2\sin^2(x)}{3cos^2(x)+3sin^2(x)}$

g. $(2\sin(x)-2\cos(x))^2$

Besides these formulas, we also have the so-called **half-angle formulas for sine, cosine and tangent**, which are derived by using the double angle formulas for sine, cosine and tangent, respectively.

Half – Angle Formulas

Using the formula $\cos(2A) = 1 - 2sin^2(A)$ and substituting $A = \frac{x}{2}$

we get $\cos(x) = 1 - 2sin^2\left(\frac{x}{2}\right)$.

Solving for $\sin\left(\frac{x}{2}\right)$, we derive the **half-angle formula for sine**:

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

Similarly, the **half-angle formula for cosine** is derived from

$\cos(x) = 2cos^2(A) - 1$, which is

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

Finally, the **half-angle formula for tangent** is:

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)} = \frac{1-\cos(x)}{\sin(x)}$$

Note: In the half-angle formulas the $\pm$ symbol is intended to mean either positive or negative but not both, and the sign before the radical is determined by the quadrant in which the half-angle $\frac{x}{2}$ terminates.

Example 9: Use a half-angle formula to find the exact value of each.

a. $\sin 15°$

b. $\cos\left(\frac{5\pi}{8}\right)$

c. $\tan\left(\frac{7\pi}{12}\right)$

Example 10: Given $\sin(x) = -\frac{4}{5}$ and $\pi < x < \frac{3\pi}{2}$, evaluate:

a) $\cos\left(\frac{x}{2}\right)$

b) $\sin\left(\frac{x}{2}\right)$

c) $\tan\left(\frac{x}{2}\right)$

Example 11. Given $\tan(x) = 2$ and $0 < x < \pi$, find $\tan(2x) + \tan\left(\frac{x}{2}\right)$.

Example 12. Given $0 < x < \pi/20$, simplify the following

$$12\cos(5x)\sqrt{\frac{1 - \cos(10x)}{2}}$$