Math.1330 – Section 6.3
Solving Trigonometric Equations

Recall algebraic equations: some of them have one solution, some of them two solutions, some of them infinitely many, and there are equations that do not have solutions at all.

Linear Equations: $2x + 5 = 1$ (One solution!)

Quadratic Equations: $x^2 + 4x - 12 = 0$ (Two solutions!)

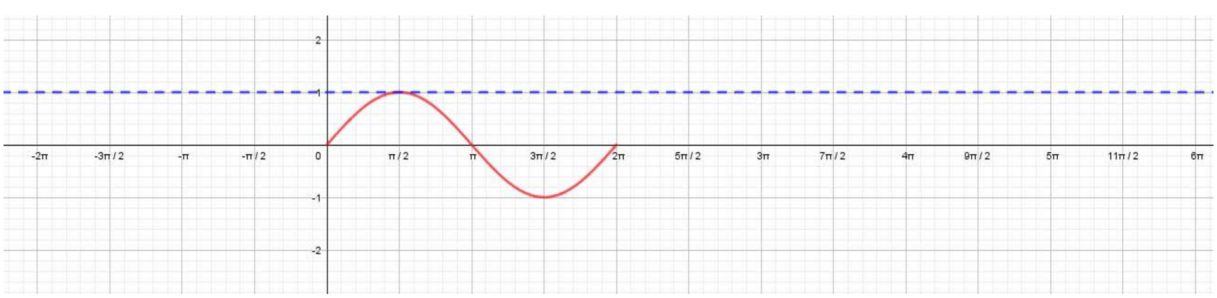
In this section, we will use all of the tools we have learned/covered in our study of trigonometry to solve *trigonometric equations*.

An equation that involves a trig function is called a trigonometric equation.
Since trigonometric functions are periodic, there may be infinitely solutions to some trigonometric equations.

Let's say we want to solve the equation: $\sin(x) = 1$

Thinking of unit circle, the first angle that comes to mind is: $\boldsymbol{x = \frac{\pi}{2}}$.
So over one period $[0,2\pi)$, this equation has only one solution.

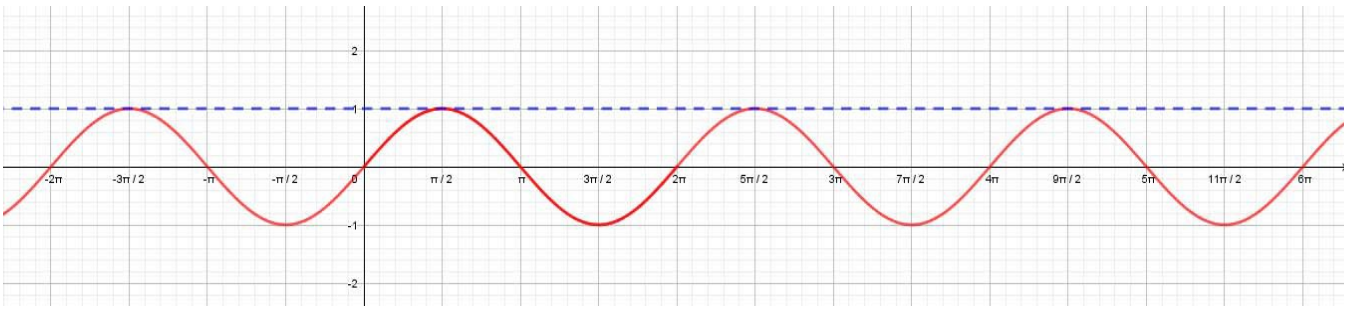
Graphically, the solution of $\sin(x) = 1$ is given by the intersection of the graph of sine with the line $y = 1$ over the interval $[0,2\pi)$:



Remember the period of the sine function is 2π, thus sine repeats itself after each rotation. Therefore, all possible solutions of this equation are:

$$x = \frac{\pi}{2} + 2k\pi, \quad \text{where } k \text{ is any integer.}$$

If you graph the big picture of sine function and look at the intersection of line $y = 1$ with the graph of sine, we get exactly the same solution set:



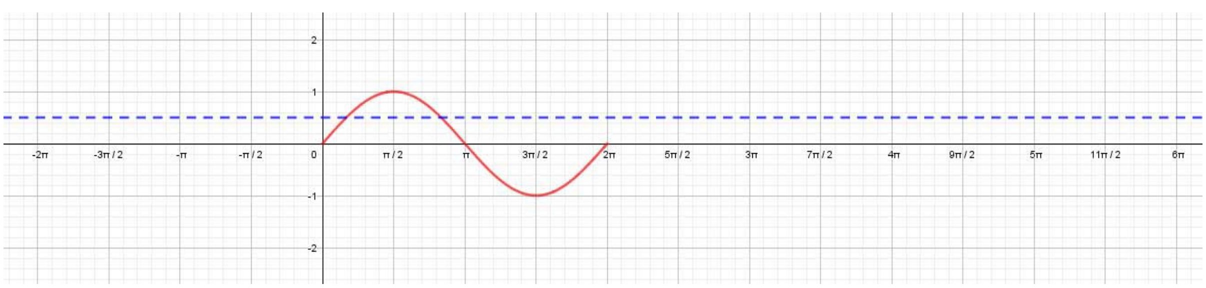
Thus, $\sin(x) = 1$ has infinitely many solutions $\left\{x = \frac{\pi}{2} + 2k\pi,\ \ k\ \ integer\right\}$.

Let's do one more example: $\sin(x) = \frac{1}{2}$

On the unit circle, the only angles we get over one period interval $[0,2\pi)$ are:

$$\boldsymbol{x = \frac{\pi}{6}} \quad \text{and} \quad \boldsymbol{x = \frac{5\pi}{6}}.$$

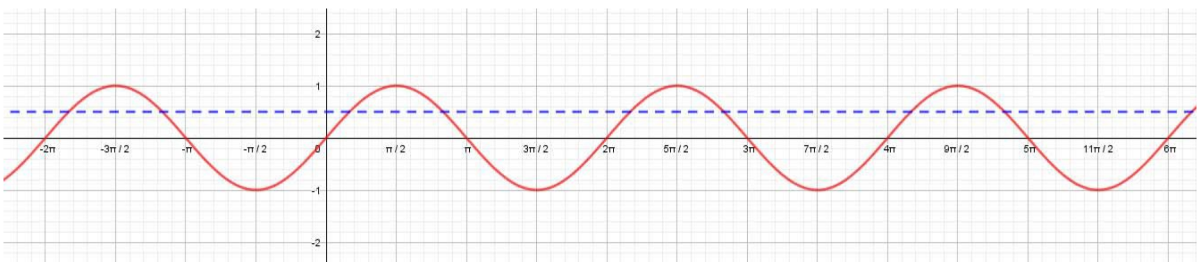
Graphically, the solution of $\sin(x) = \frac{1}{2}$ is given by the intersection of the graph of sine with the line $y = \frac{1}{2}$ over the interval $[0,2\pi)$:



Since the period of the sine function is 2π, then all possible solutions of this equation are:

$$x = \frac{\pi}{6} + 2k\pi \quad and \quad x = \frac{5\pi}{6} + 2k\pi, \qquad where\ k\ is\ an\ integer.$$

If you graph the big picture of sine function and look at the intersection of line $y = \frac{1}{2}$ with the graph of sine, we get exactly the same solution set:



Again, there are infinitely many solutions to this equation.

Recall: For sine and cosine functions, the period is 2π .

For tangent and cotangent functions, the period is π .

Example 1:

a) Solve the equation in the interval $[0, \pi)$: $\tan(x) = -1$

b) Find all solutions to the equation: $\tan(x) = -1$

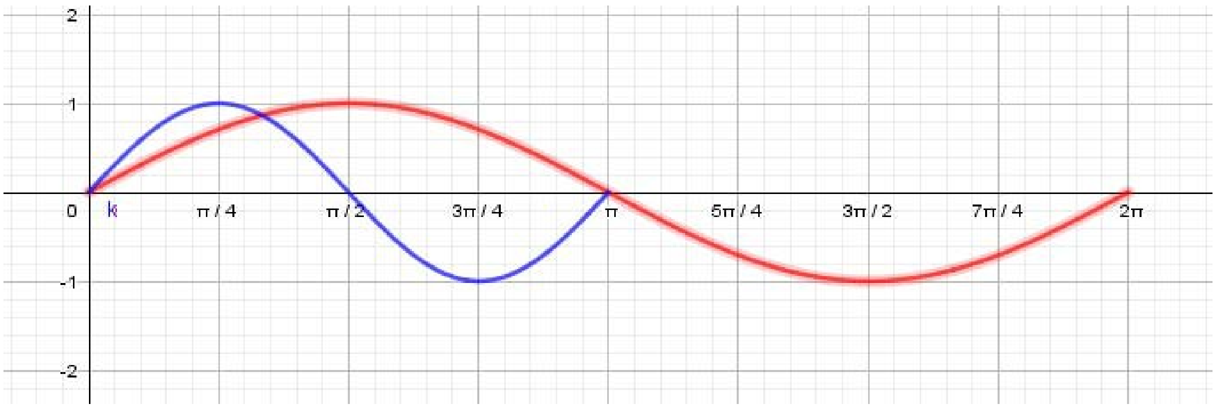
Example 2:

a) Solve the equation in the interval $[0, 2\pi)$: $2\cos(x) + 4 = 3$

b) Find all solutions to the equation: $2\cos(x) + 4 = 3$

Remark: If the trigonometric function used in an equation is of the form $\sin(Bx)$, $\cos(Bx)$ and/or $\tan(Bx)$, then the period will change accordingly.

Here are the graphs of $f(x) = \sin(x)$ and $g(x) = \sin(2x)$ over one period:



The solution to $sin(x) = 1$ is $x = \frac{\pi}{2}$.

The solution to $\boldsymbol{sin(2x) = 1}$ is $2x = \frac{\pi}{2} \rightarrow \boldsymbol{x = \frac{\pi}{4}}$.

Example 3: Solve the equation in the interval $[0, \pi)$: $\sqrt{2}\sin(2x) - 1 = 0$

Example 4: Find all the solutions to the equation: $\cos(4x) + 1 = 1$.

Example 5

a) Find all solutions to the equation in the interval $[0,4\pi)$: $2\sin\left(\frac{x}{2}\right) = -\sqrt{3}$

b) Find all solutions to the equation in the interval $[0,8\pi)$: $2\sin\left(\frac{x}{2}\right) = -\sqrt{3}$

Example 6. Solve the equation in the interval $[0,2)$: $\cot(\pi x) = 1$

Example 7. Find all solutions of the equation in the interval $[0, \pi)$:

$2\cos\left(2x - \frac{\pi}{4}\right) = \sqrt{2}$

Example 8. Find all solutions of the equation $4\sin(2x) + 1 = 2$

Example 9. Find all solutions of the equation in the interval $[0,2\pi)$:

$2\sin^2(x) + 1 = 3$

Example 10. Find all solutions of the equation in the interval $[0,2\pi)$:

$2\sin^2(x) - 5\sin(x) - 3 = 0$

Example 11. Find all solutions of the equation in the interval $[0,2\pi)$:

$\tan^3(x) - \tan(x) = 0$

Example 12. Find all solutions of the equation in the interval $[0,2\pi)$:

$\csc^2(x) = 4$

Example 13. Find all solutions of the equation in the interval $[0,2\pi)$:

$\sin^2(x)\cos(\mathrm{x}) = \cos(x)$

Example 14: Let $f(x) = cos^2(x) - sin^2(x)$.

Find the x-intercepts of this function over the interval $[0,2\pi)$.

Example 15. Without solving the equation, determine the number of the solutions on the given interval:

a) $\sin(x) = 1$ over $[0,2\pi)$ → Number of solutions:____________.

b) $\sin(x) = \frac{1}{4}$ over $[0,2\pi)$ → Number of solutions:____________.

c) $\sin(x) = -\frac{1}{5}$ over $[0,2\pi)$ → Number of solutions:____________.

d) $\sin(x) = -1$ over $[0,2\pi)$ → Number of solutions:____________.

e) $\sin(x) = 2$ over $[0,2\pi)$ → Number of solutions:____________.

f) $\sin(x) = 1$ over $[0,4\pi)$ → Number of solutions:____________.

g) $\sin(x) = \frac{1}{4}$ over $[0,4\pi)$ → Number of solutions:____________.

h) $\sin(x) = 0$ over $[0,4\pi)$ → Number of solutions:____________.

i) $\sin(x) = -5$ over $[0,4\pi)$ → Number of solutions:____________.

Example 16. Determine the number of the solutions OVER ONE PERIOD.

a) $2\cos(x) = 1$ → Number of solutions:____________.

b) $4\cos(2x) + 1 = 2$ → Number of solutions:____________.

c) $4\cos(2x) + 2 = 2$ → Number of solutions:____________.

Example 17. Determine the number of the solutions On the Number Line.

a) $\cos(x) = \frac{1}{2}$ → Number of solutions:____________.

b) $2\cos(x) = 4$ → Number of solutions:____________.