Math.1330 – Section 7.3
The Law of Sines and the Law of Cosines

A triangle that is not a right triangle is called **an oblique triangle**. To solve an oblique triangle we will not be able to use right triangle trigonometry. Instead, we will use the Law of Sines and/or the Law of Cosines.

Solving a triangle means finding all three angles and all three sides. We will typically be given three parts of the triangle and we will be asked to find the other three. The approach we will take to the problem will depend on the information that is given.

If you are given SSS (the lengths of all three sides) or SAS (the lengths of two sides and the measure of the included angle), you will use the Law of Cosines to solve the triangle.

If you are given SAA (the measures of two angles and one side) or SSA (the measures of two sides and the measure of an angle that is not the included angle), you will use the Law of Sines to solve the triangle.

Recall from your geometry course that SSA does not necessarily determine a triangle. We will need to be careful when this is the given information.

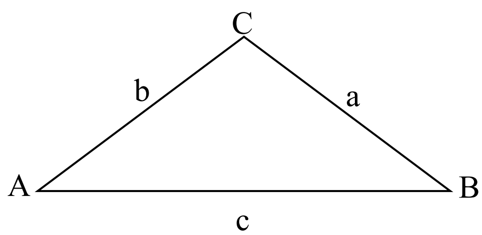
Since you will have three pieces of information when solving a triangle, it is possible for you to use both the Law of Sines and the Law of Cosines in the same problem.

When drawing a triangle, follow some basic geometric rules:

- the measure of the largest angle is opposite the longest side;
- the measure of the middle-sized angle is opposite the middle-sized side;
- the measure of the smallest angle is opposite the shortest side.

THE LAW OF COSINES (used for SSS, SAS cases)

If we are given SSS or SAS, you will use **the Law of Cosines** to solve the triangle. Here's the **Law of Cosines i**n any triangle *ABC,*



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example 1: In ΔABC, $a = 2$, $c = 5$, and $b = \sqrt{39}$. Find the measure of $\angle B$.

Example 2: In ΔABC, $\angle B = 60°$, $a = 17$ and $c = 12$. Find the length of AC.

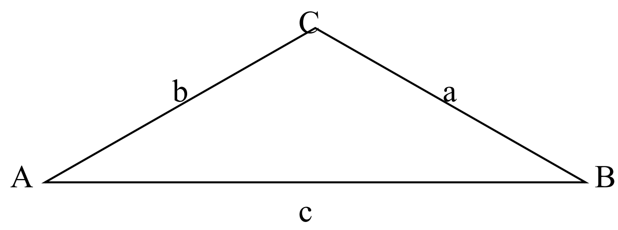
Example 3: Two sailboats leave the same dock together traveling on courses that have an angle of 135° between them. If each sailboat has traveled 3 miles and 5 miles separately, how far apart are the sailboats from each other?

Example 4: An isosceles triangle has a vertex angle of $150°$. If the equal sides are 10 units each, find the length of the base.

Example 5. In triangle $ABC, \angle B = 50°,\ a = 10\ \ and\ \ c = 12$. Find AC. (You might need a calculator for this problem.)

THE LAW OF SINES (used for SAA, SSA cases)

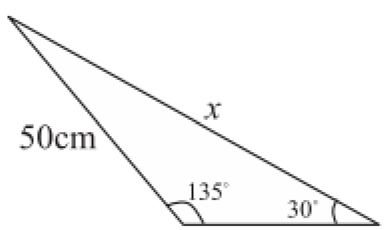
If we are given SAA or SSA, we will use **the Law of Sines** to solve the triangle. Here's the **Law of Sines i**n any triangle *ABC,*



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

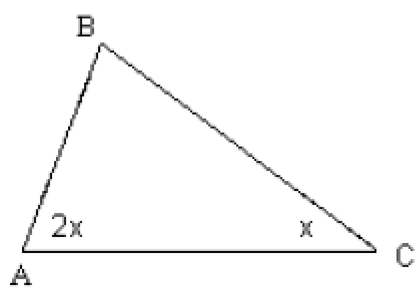
Recall from your geometry course that SSA does not necessarily determine a triangle. We will need to be careful when this is the given information.

Example 6: Find x.



Example 7: In ΔXYZ, $\angle X = 26°$, $\angle Z = 78°$ and $y = 18$. Solve the triangle.

Example 8. In the triangle below, $AB = 9,\ BC = 12$. Find the value of x.
(You may use inverse trig notation to express your answer.)



Example 9. In ΔABC, $\angle A = 30°,\ b = 6\sqrt{2}\ and\ a = 6$. Find measures of angles B and C.

Example 10. In ΔABC, $\angle A = 60°,\ a = 8\sqrt{3}\ and\ b = 8\sqrt{2}$. Find all possible measures for angle B.

Example 11: In $\Delta ABC, \angle A = 50°, b = 9$ and $a = 6$. Find AB.

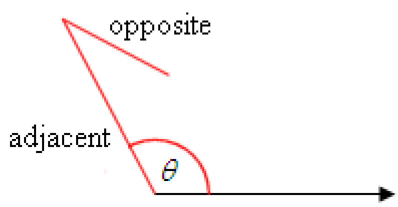
***Note: SSA* case is called the ambiguous case of the law of sines.**
There may be two solutions, one solution, or no solutions.
You should throw out the results that don't make sense.
That is, if $\sin A > 1$ or the angles add up to more than 180^0.

SSA Case (Two sides and an angle opposite to those sides)

In the last case (SSA) for solving oblique triangles, two sides and the angle opposite one of those sides are given. Suppose that θ is the given angle. The other given side must be adjacent to θ. We consider several cases.

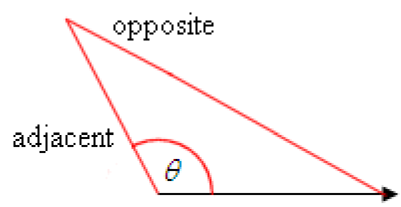
Case 1: Suppose that $\theta > 90^\circ$. Two possibilites arise.
(a) If opposite $\leq$ adjacent, no triangle is formed. (There is no solution.)
(b) If opposite > adjacent, one triangle is formed. (Use the Law of Sines.)

Case 1 Part (a)



$\theta > 90^\circ$
opposite $\leq$ adjacent

Case 1 Part (b)



$\theta > 90^\circ$
opposite > adjacent

Case 2: Suppose that $\theta < 90^\circ$. Four possibilites arise. Let h be the length of the altitude of the triangle drawn from the vertex that connects the opposite and adjacent sides.

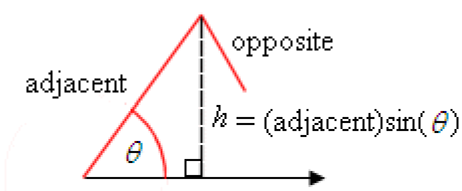
(a) If opposite $<h<$ adjacent, no triangle is formed. (There is no solution.)

(b) If opposite $=h<$ adjacent, one right triangle is formed. (Use right triangle trigonometry to solve the triangle as in Section 7.1.)

(c) If $<h<$ opposite $<$ adjacent, two different triangles are formed. This is called the ambiguous case. (Use the Law of Sines to find two solutions.)

(d) If opposite $\geq$ adjacent, one triangle is formed. (Use the Law of Sines.)

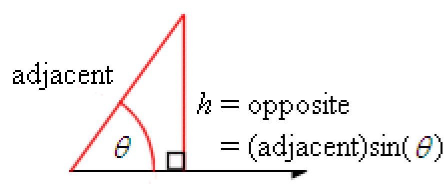
Case 2 Part (a)



$\theta < 90^\circ$

opposite $<h<$ adjacent

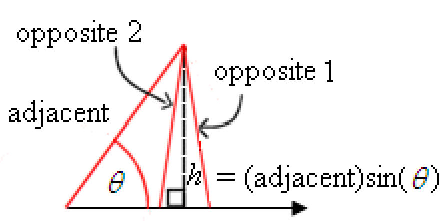
Case 2 Part (b)



$\theta < 90^\circ$

opposite $=h<$ adjacent

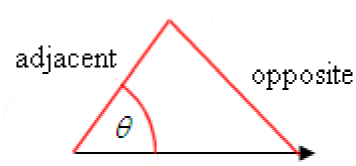
Case 2 Part (c)



$\theta < 90^\circ$

$h<$ opposite $<$ adjacent

Case 2 Part (d)



$\theta < 90^\circ$

opposite $\geq$ adjacent

Example 12: In ΔPQR, $\angle P = 112°$, $p = 5$ and $q = 7$.
How many possible triangles are there? Solve the triangle.
Round the answers to three decimal places.

Example 13: In ΔXYZ, $\angle Y = 22°$, $y = 7$, $x = 5$.
How many possible triangles are there?
Solve the triangle and round all answers to the nearest hundredth.

Example 14. Given triangle ABC, $AB = 15, BC = 5\sqrt{3}, \hat{A} = 30^\circ$.
How many choices are there for the measure of angle C?

Example 15. Given triangle ABC, $AB = 3, BC = \sqrt{2}, \hat{A} = 45^\circ$.
How many choices are there for the measure of angle C?