Popper $\underline{\underline{03}}$

① $f(x) = -2(x-3)^1(x+4)^1(2-x)^1 \longleftrightarrow$ example 2

How does the end behaviour of graph look like)

A. $\nwarrow \ldots \nearrow$ B. $\swarrow \ldots \searrow$ C. $\swarrow \ldots \nearrow$ D. $\nwarrow \ldots \searrow$

degree $f = 1+1+1 = 3$ (odd)

leading coefficient $= (-2)\cdot(1)^1\cdot(1)^1\cdot(-1)^1 = +2 = (+)$ $\Big\}$ $\swarrow \ldots \nearrow$

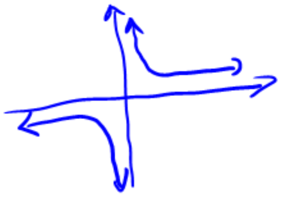
② $f(x) = -2(x-1)^2\cdot(x+4)(x+5)^3 \longleftrightarrow$ example 3.

Find leading term of the polynomial!

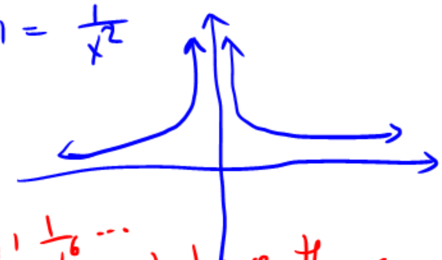
A. $-2x^6$ B. $2x^6$ C. x^6 D. none

$-2\cdot x^2\cdot x^1\cdot x^3 = -2x^6$

Recall $f(x)=\frac{1}{x}$. Also $f(x)=\frac{1}{x^3}, \frac{1}{x^5} \cdots$ behave the same



$f(x)=\frac{1}{x^2}$ Also $f(x)=\frac{1}{x^4}, \frac{1}{x^6} \cdots$ behave the same.



Math 1330 - Section 2.3
Rational Functions

A rational function is a function of the form $f(x)=\dfrac{P(x)}{Q(x)}$, where P and Q are polynomial functions and $Q(x)\neq 0$. You'll need to be able to find the following features of the graph of a rational function and then use the information to sketch the graph.

- Domain
- Intercepts
- Holes
- Vertical asymptotes
- Horizontal asymptote
- Slant asymptote
- Behavior near the vertical asymptotes

ex. $f(x)=\dfrac{x^2-2x}{x^2-4}$ rational

Domain: The domain of f is all real numbers except those values for which $Q(x)=0$.

$f(x)=\dfrac{x^2-2x}{x^2-4}$

$x^2-4=0$
$(x-2)(x+2)=0$
$x=2,-2$

$\Rightarrow$ domain $= (-\infty,-2)\cup(-2,2)\cup(2,\infty)$

***x* intercept(s):** The x intercept(s) of the function will be all values of x for which $P(x)=0$, but $Q(x)\neq 0$. Before finding any x-int., factor!!!

$f(x)=\dfrac{x^2-2x}{x^2-4}=\dfrac{x\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$

If some factor simplified, then it can't be x-int. It is a hole.

$\Rightarrow x=0$ is the x-intercept.

***y* intercept:** The y intercept of the function is $f(0)$.

$\hookrightarrow y=f(0)=\dfrac{0^2-2\cdot 0}{0^2-4}=0$ $\leftarrow$ just substitute zero in x.

Holes: The graph of the function will have a hole at any value of x for which both $P(x)=0$ and $Q(x)=0$.

$f(x)=\dfrac{x\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$

$f^*(x)=\dfrac{x}{x+2}$

If a factor is simplified from both $P(x)$ & $Q(x)$, then it is a hole.

$\Rightarrow x-2=0 \Rightarrow x=2$ a hole

Location of this hole is in the graph at $f^*(2)=\dfrac{2}{2+2}=\dfrac{1}{2} \Rightarrow \boxed{(2,\tfrac{1}{2})}$

Vertical asymptotes: The graph of the function has a vertical asymptote at any value of x for which $Q(x)=0$ but $P(x)\neq 0$.

$$f(x)=\frac{x\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}=\underbrace{\frac{x}{x+2}}$$

First, always simplify the <u>hole</u> factors.

$$x+2=0 \Rightarrow \boxed{x=-2}$$

Horizontal asymptote: You can determine if the graph of the function has a horizontal asymptote by comparing the degree of the numerator with the degree of the denominator.

$$f(x)=\frac{1\cdot x^2-2x}{1\cdot x^2-4} \xrightarrow[\text{run}]{\text{big}} \frac{x^2}{x^2}=1 \Rightarrow y=\frac{1}{1}=1$$

There are three cases

I. if the degree of the numerator is smaller than the degree of the denominator, then the graph of the function has a horizontal asymptote at $y=0$.

ex. $$f(x)=\frac{x+4}{x^3+2x-2} \xrightarrow[\text{run}]{\text{big}} \frac{x}{x^3} \to \frac{1}{\underbrace{x^2}_{\text{very small}}} \to \boxed{y=0 \text{ H.A.}}$$

II. if the degree of the numerator is equal to the degree of the denominator, then the graph of the function has a horizontal asymptote at

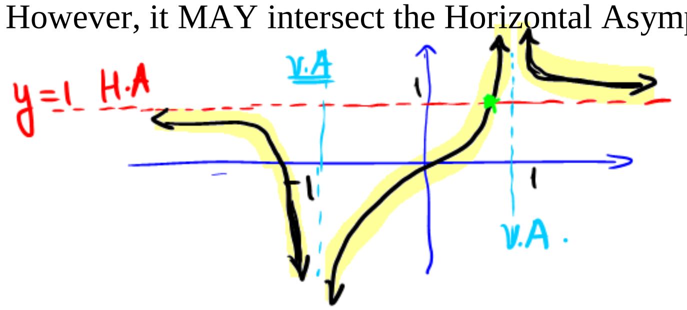
$$y=\frac{\text{leading coefficent of numerator}}{\text{leading coefficient of denominator}}$$

ex. $$f(x)=\frac{3x^2-5}{4x^2+2x-1} \xrightarrow[\text{run}]{\text{big}} \frac{3\cancel{x^2}}{4\cancel{x^2}} \to \boxed{\frac{3}{4}=y \quad \text{H.A.}}$$

III. if the degree of the numerator is greater than the degree of the denominator, then the graph of the function does not have a horizontal asymptote.

ex. $$f(x)=\frac{x^8+2x-3}{100x^2+2} \xrightarrow[\text{run}]{\text{big}} \frac{\cancel{x^8}}{100\cancel{x^2}} \to \underbrace{\frac{x^6}{100}}_{\text{very big}} \Rightarrow \boxed{\text{No H.A.}}$$

Note: The graph of a function $f(x)$ can never intersect the Vertical Asymptote. However, it MAY intersect the Horizontal Asymptote.



Look at highlighted graph. This graph does not touch V.A., but it touches the H.A.

To be continued on Monday, 02/08

Example: Given $f(x) = \dfrac{x-2}{x^2-1}$, find the point at which $f(x)$ intersects the HA.

First, find HA

$\Rightarrow y = 0$, then take $f(x) = 0$ ← HA value

$\dfrac{x-2}{x^2-1} = 0$ iff $x - 2 = 0$

$\boxed{x = 2}$

Example: Given $f(x) = \dfrac{x^2-2x+2}{x^2-x}$, find the point at which $f(x)$ intersects the HA.

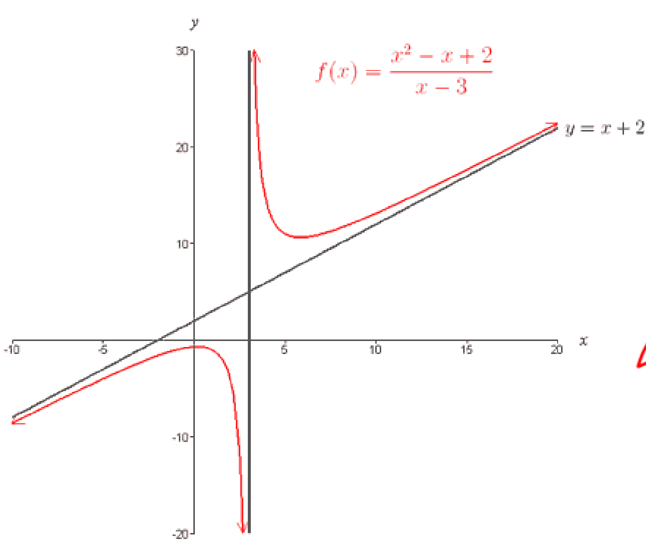
First, find HA $\Rightarrow y = \frac{1}{1} = 1$.

Then, say $f(x) = 1$ i.e. $\dfrac{x^2-2x+2}{x^2-x} = 1$

$\Rightarrow \cancel{x^2} - 2x + 2 = \cancel{x^2} - x$

$\Rightarrow \boxed{x = 2}$

Slant asymptote: The graph of the function may have a slant asymptote if the degree of the numerator is greater than the degree of the denominator. To find the equation of the slant asymptote, use long division to divide the denominator into the numerator. The quotient is the equation of the slant asymptote.



Slant asymptotes occur for $f(x) = \dfrac{P(x)}{Q(x)}$, when deg $P(x)$ is one bigger than deg $Q(x)$.

Long Division: Quotient = slant

$$\begin{array}{r} x + 2 \\ x-3 \overline{) x^2 - x + 2} \\ -(x^2 - 3x) \\ \hline 2x + 2 \\ -(2x - 6) \\ \hline 8 \end{array}$$

Remainder

$f(x) = (x+2) + \dfrac{8}{x-3}$ ⇐

Examples: Practice Long Division to get these results:

$$f(x)=\frac{x^2+4}{x}=x+\frac{4}{x}$$; the slant asymptote is: $y=x$.

$$f(x)=\frac{2x^2+1}{x}=2x+\frac{1}{x}$$; the slant asymptote is: $y=2x$.

$$f(x)=\frac{x^2-2}{x-1}=x+1-\frac{1}{x-1}$$; the slant asymptote is: $y=x+1$.

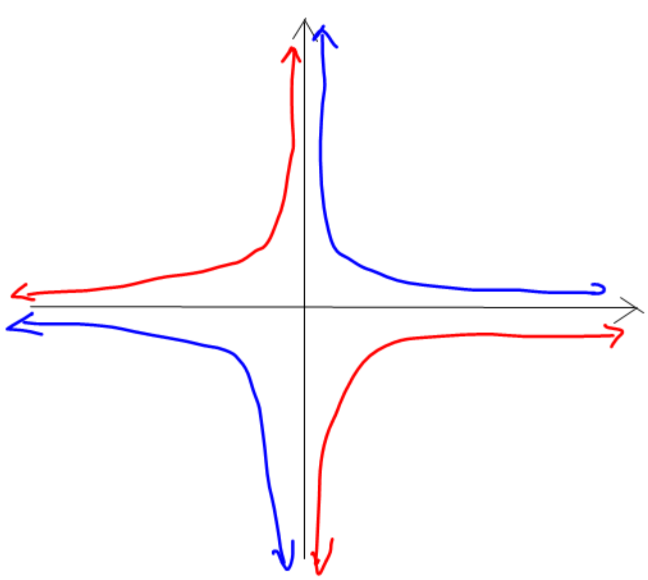
$$f(x)=\frac{x^2+x}{x-2}=\text{Quotient}+\frac{\text{Remainder}}{x-2}$$; the slant asymptote is: $y=\mathit{Quotient}$.

ALWAYS → FOLLOW THE FLOW of FUNCTION!

Behavior near the vertical asymptotes: The graph of the function will approach either ∞ or $-\infty$ on each side of the vertical asymptotes. To determine if the function values are positive or negative in each region, find the sign of a test value close to each side of the vertical asymptotes.

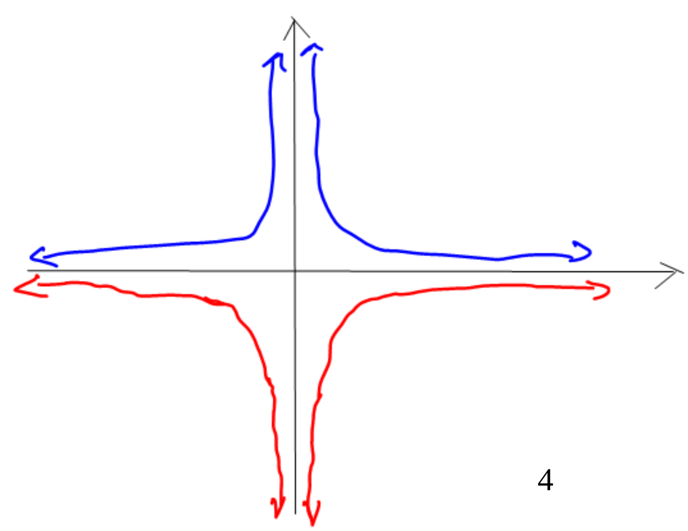
$f(x)=\frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^5} \cdots$ (same shape)

$f(x)=\frac{-1}{x}, \frac{-1}{x^3}, \frac{-1}{x^5}$ (same shape)



$f(x)=\frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^6} \cdots$

$f(x)=\frac{-1}{x^2}, \frac{-1}{x^4}, \frac{-1}{x^6} \cdots$



Example 1: Sketch the graph of: of $f(x) = \dfrac{2x^2-8}{x^2-3x+2}$.

Factor $= \dfrac{2(x^2-4)}{(x-1)(x-2)} = \dfrac{2(x+2)\cancel{(x-2)}}{(x-1)\cancel{(x-2)}}$

- domain: $x \neq 1, x \neq 2 \Rightarrow (-\infty,1)\cup(1,2)\cup(2,\infty)$

- At $x=2$, graph has a hole

  $f^*(x) = \dfrac{2(x+2)}{x-1} \Rightarrow f(2) = \dfrac{2(2+2)}{2-1} = 8$

  $\boxed{(2,8)}$ hole

- x-int $\Rightarrow x+2=0$

  $x=-2$ $\boxed{(-2,0)}$ x-int

- y-int $\Rightarrow f(0) = \dfrac{2(0+2)}{0-1} = -4$

  $\boxed{(0,-4)}$ y-int

- V.A. $x-1=0$

  $\boxed{x=1}$ V.A.

- H.A. $\Rightarrow \boxed{y=\frac{2}{1}=2}$ H.A.

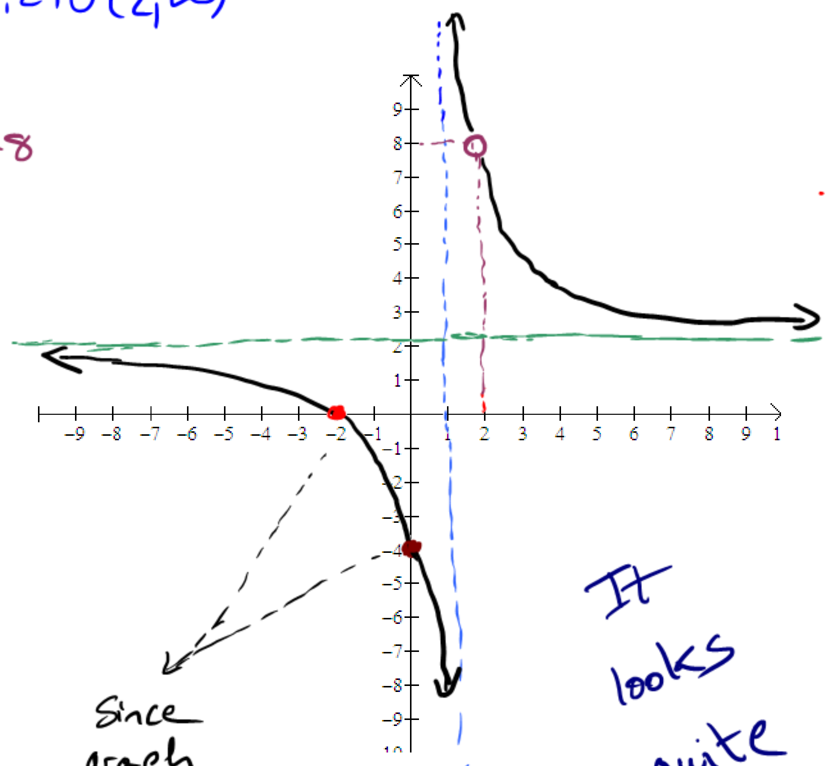
Does f cut H.A $y=2$? No!

$f(x) = \dfrac{2(x+2)}{x-1} = 2 \Rightarrow \boxed{\text{no solution}}$

Solve:

$$\frac{2(x+2)}{x-1} = \frac{2}{1} \quad \text{cross-product}$$

$$\cancel{2}(x+2)\cdot 1 = \cancel{2}(x-1)$$
$$x+2 = x-1$$
$$2 \neq -1$$



Since graph passes through those points then it should be on this side

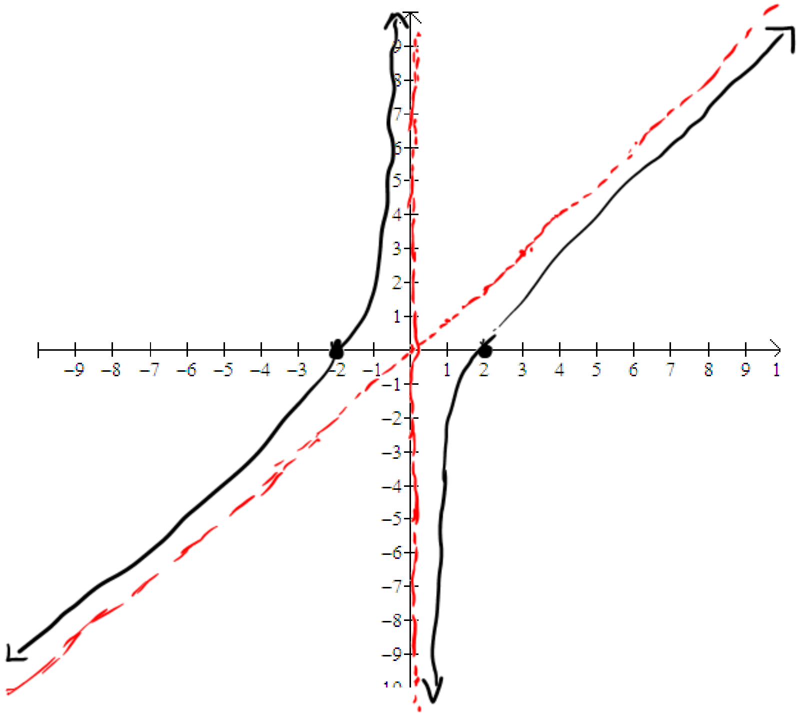
It looks quite fancy

(Sometimes, you may skip long division)

Example 2: Sketch the graph of: $f(x)=\dfrac{x^4-16}{x^3}$. $=\dfrac{x^4}{x^3}-\dfrac{16}{x^3}=x-\dfrac{16}{x^3}$

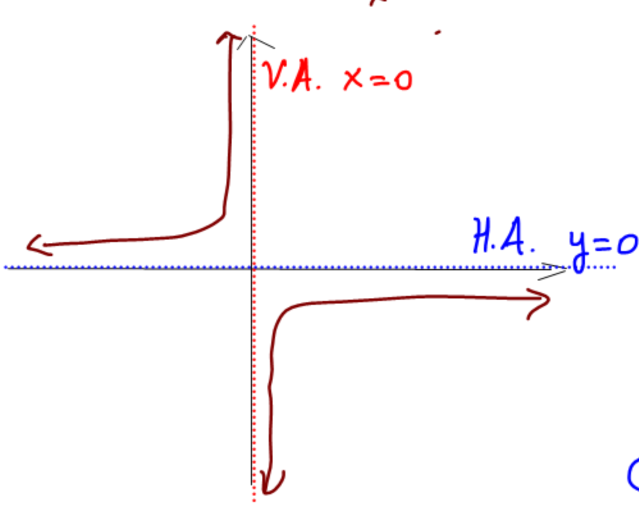
- Long division

$$x^3 \overline{)\,x^4-16}\quad\text{quotient: } x$$
$$-x^4$$
$$-16$$



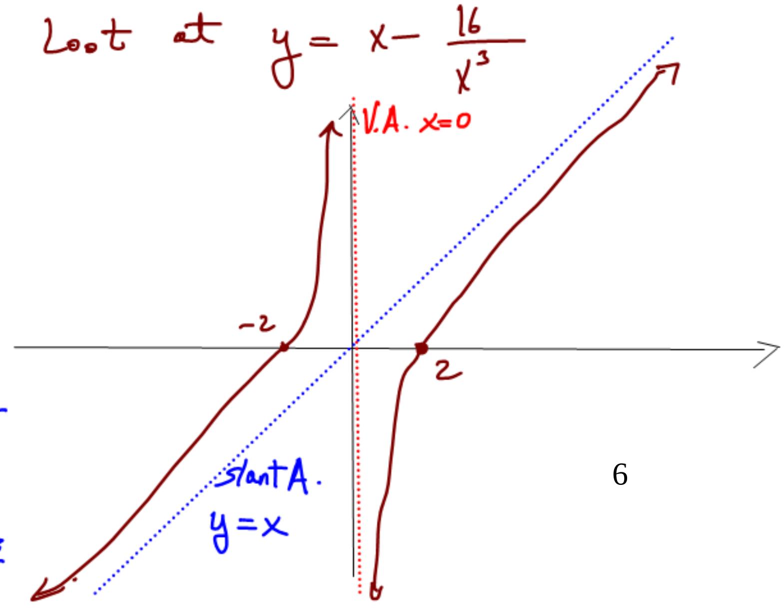
- Slant asymptote: $y=x$
- domain: $x\neq 0$
  $(-\infty,0)\cup(0,\infty)$
- V.A: $x=0$
- H.A. none
- x-int.: $f(x)=0$
  $\dfrac{x^4-16}{x^3}=0 \Rightarrow x^4-16=0$
  $x=\pm 2$
- y-int: none
- holes: none

Look at $y=\dfrac{-16}{x^3}$



NOTICE $\Longrightarrow$ The CHANGE

Look at $y=x-\dfrac{16}{x^3}$



Exercise (Do on your own)

Exercise: Find all of the features of $f(x)=\dfrac{x^3-4x^2}{x^2-2x-8}$ and use them to graph the function.

$$f(x)=\frac{x^2\cancel{(x-4)}}{\cancel{(x-4)}(x+2)}=\frac{x^2}{x+2}$$

$\Rightarrow$ domain: $x\neq 4,\ x\neq -2$

$(-\infty,-2)\cup(-2,4)\cup(4,\infty)$

$\Rightarrow$ f has a hole at $x=4$

$$f^*(4)=\frac{4^2}{4+2}=\frac{16}{6}=\frac{8}{3}$$

$\Rightarrow$ x-int: $f^*(x)=0$

$$\frac{x^2}{x+2}=0 \Rightarrow x^2=0$$

$x=0$ parabola

$\Rightarrow$ y-int: $y=f(0)=\dfrac{0^2}{0+2}=0$

$\Rightarrow$ V.A. $x+2=0 \Rightarrow \boxed{x=-2}$

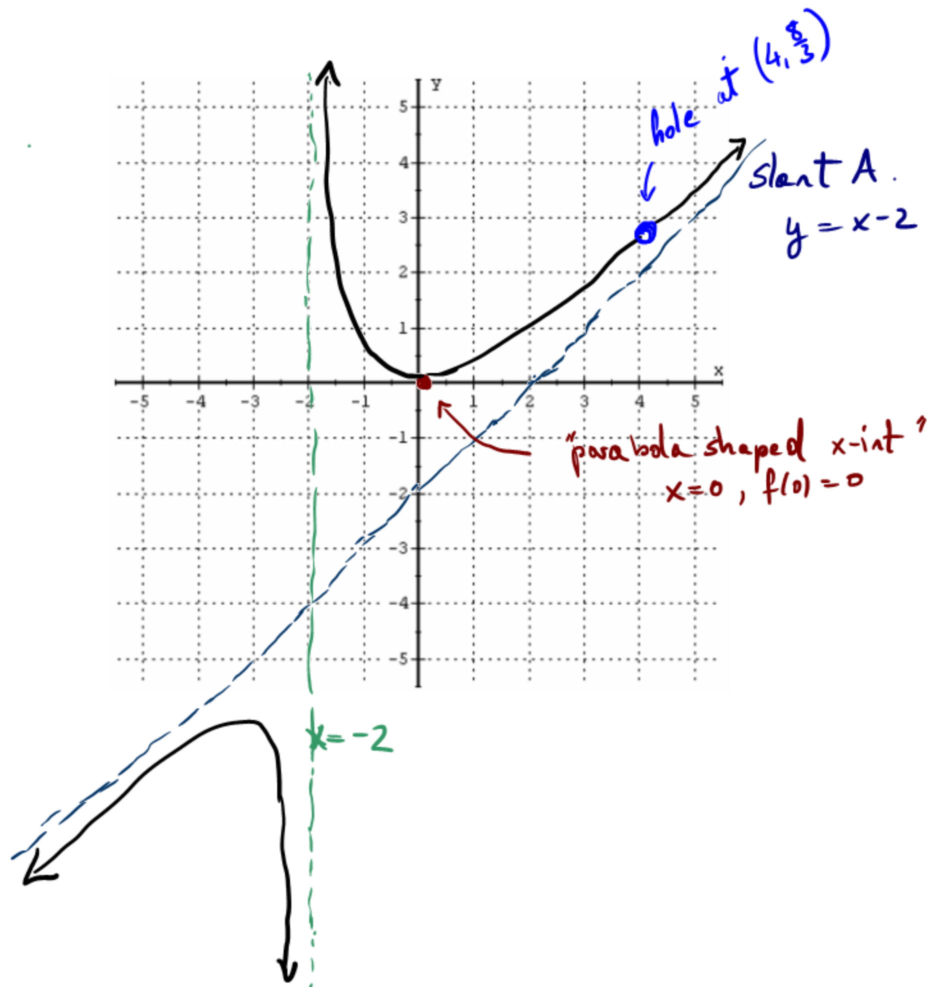
$\Rightarrow$ H.A. none

$\Rightarrow$ Slant A. $f(x)=\dfrac{x^2}{x+2}=x-2+\dfrac{4}{x+2}$

Long division

$$\begin{array}{r|l} & x-2 \;=y \\ x+2 & x^2 \\ & -(x^2+2x) \\ \hline & -2x \\ & -(-2x-4) \\ \hline & 4 \end{array}$$

$\boxed{y=x-2}$



Another fancy graph!