

## Popper #24

IV,  $60^\circ \rightarrow \sin 60^\circ = \frac{\sqrt{3}}{2}$

①  $\sin(300^\circ) = -\frac{\sqrt{3}}{2}$

A.  $\frac{1}{2}\sqrt{3}$     B.  $-\sqrt{3}$     C.  $-\frac{1}{2}\sqrt{3}$     D.  $\frac{1}{2}$     E.  $-\frac{1}{2}$

IV,  $30^\circ \rightarrow \tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

②  $\tan\left(\frac{11\pi}{6}\right) = \tan(330^\circ) = -\frac{\sqrt{3}}{3}$

A.  $\frac{1}{3}\sqrt{3}$     B.  $\frac{1}{3}\sqrt{3}$     C.  $-\frac{1}{3}\sqrt{3}$     D.  $-\sqrt{3}$     E.  $\sqrt{3}$

③  $\cot(2\pi) = \frac{\cos(2\pi)}{\sin(2\pi)} = \frac{1}{0} = \text{undefined}$

A. undefined    B. -1    C. 0    D. 1    E.  $\sqrt{3}$

④  $\sin\left(-\frac{7\pi}{6}\right) = \sin\left(-210^\circ\right) = +\sin(30^\circ) = +\frac{1}{2}$   
reference

A.  $-\frac{1}{2}$     B.  $\frac{\sqrt{3}}{3}$     C.  $\frac{1}{2}$     D.  $-\frac{1}{2}\sqrt{3}$     E.  $\frac{1}{2}\sqrt{3}$

## Section 4.4

### Trigonometric Expressions and Identities

① use identities

② do basic math operations

In this section, you'll learn to simplify trig expressions using identities and using basic algebraic operations. You can add, subtract, multiply, divide and factor trig expressions, in much the same manner than you can with algebraic expressions.

Notation:  $(\sin \theta)^n = \sin^n \theta$

eg.  $(\sin \theta)^3 = \sin^3 \theta$   
 $(\tan \theta)^2 = \tan^2 \theta$ .

**Example 1:** Perform the following operation and simplify:

$$(\cos(\theta) + 5)(\cos(\theta) - 7)$$

$$= \cos^2 \theta - 7 \cos \theta + 5 \cos \theta - 35$$

$$= \cos^2 \theta - 2 \cos \theta - 35.$$

Recall FOIL process

$$(a + 5)(a - 7)$$

$$= a^2 - 7a + 5a - 35$$

$$= a^2 - 2a - 35$$

**Example 2:** Factor:  $\sin^2 x - \sin x - 2$

$$\sin^2 x - \sin x - 2$$

$$= (\sin x - 2)(\sin x + 1).$$

$$y^2 - y - 2$$
$$= (y - 2)(y + 1)$$

**Example 3:** Factor:  $\sec^2(\theta) - 4\sec(\theta) - 12$

$$\begin{aligned} & \sec^2 \theta - 4 \sec \theta - 12 \\ &= (\sec \theta - 6)(\sec \theta + 2) \end{aligned}$$

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$$\begin{aligned} 12 &= \cancel{3 \times 4} \\ &= \boxed{6 \times 2} \\ -6 + 2 &= \underline{\underline{-4}} \end{aligned}$$

Sometimes, you can use **trig identities** to help you simplify trig expressions. Here is a list of trig identities we have already met. Note that NONE of these identities will be provided on the tests. You must know all of these identities.

$$\tan(t) = \frac{\sin(t)}{\cos(t)}$$

$$\cot(t) = \frac{\cos(t)}{\sin(t)}$$

### Reciprocal Identities:

$$\begin{array}{l} \sin t \rightarrow \\ \cos t \rightarrow \\ \tan t \rightarrow \end{array} \left\{ \begin{array}{l} \csc(t) = \frac{1}{\sin(t)}, \sin(t) \neq 0 \\ \sec(t) = \frac{1}{\cos(t)}, \cos(t) \neq 0 \\ \cot(t) = \frac{1}{\tan(t)}, \tan(t) \neq 0 \end{array} \right.$$

PRACTICE,

LEARN,

REMEMBER !!!

### Pythagorean Identities

$$\sin^2(t) + \cos^2(t) = 1$$

$$1 + \tan^2(t) = \sec^2(t)$$

$$1 + \cot^2(t) = \csc^2(t)$$

You should know all three of the Pythagorean Identities or be able to derive the last two from the first one.

### Opposite Angle Identities

$$\sin(-t) = -\sin(t)$$

$$\cos(-t) = \cos(t)$$

$$\tan(-t) = -\tan(t)$$

$$\csc(-t) = -\csc(t)$$

$$\sec(-t) = \sec(t)$$

$$\cot(-t) = -\cot(t)$$

We'll use these in the next several examples.

**Example 4:** Simplify  $\tan(-x)\csc(x) = \boxed{-\sec x}$

$$\begin{aligned}\tan(-x) \cdot \csc(x) &= -\tan x \cdot \csc x \\ &= -\frac{\cancel{\sin x}}{\cos x} \cdot \frac{1}{\cancel{\sin x}} \\ &= \frac{-1}{\cos x} = \boxed{-\sec x}\end{aligned}$$

**Example 5:** Simplify:  $[\csc(x) - 1][\sec(x) + \tan(x)] \leftarrow \text{Do FOIL}$

$$= \csc x \cdot \sec x + \underbrace{\csc x \cdot \tan x}_{\text{by above} = \sec x} = \sec x - \tan x$$

$$= \csc x \cdot \sec x + \cancel{\sec x} - \cancel{\sec x} - \tan x$$

$$= \csc x \cdot \sec x - \tan x$$

$$= \frac{1}{\sin x} \cdot \frac{1}{\cos x} - \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x}$$

$$= \frac{1 - \sin^2 x}{\cos x \cdot \sin x}$$

$$= \frac{\cos^2 x}{\cos x \cdot \sin x} = \frac{\cancel{\cos x} \cdot \cos x}{\cancel{\cos x} \cdot \sin x}$$

$$= \boxed{\cot x}$$

We know:

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow 1 - \sin^2 x = \cos^2 x$$

To be continued on Monday, 04/04

**Example 6:** Simplify:  $[1 - \cos(x)][\csc(x) + \cot(x)] = \boxed{\sin x}$

$$[1 - \cos x] \cdot [\csc x + \cot x]$$

$$= (1 - \cos x) \left( \frac{1}{\sin x} + \frac{\cos x}{\sin x} \right)$$

$$= (1 - \cos x) \left( \frac{1 + \cos x}{\sin x} \right)$$

$$= \frac{(1 - \cos x)(1 + \cos x)}{\sin x} = \frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \boxed{\sin x}$$

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Or :

$$[1 - \cos x][\csc x + \cot x]$$

$$= \csc x + \cot x - \cos x \cdot \csc x - \cos x \cdot \cot x$$

$$= \frac{1}{\sin x} + \frac{\cancel{\cos x}}{\cancel{\sin x}} - \cancel{\cos x} \cdot \frac{1}{\sin x} - \cos x \cdot \frac{\cos x}{\sin x}$$

$$= \frac{1}{\sin x} - \frac{\cos^2 x}{\sin x} = \frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \boxed{\sin x}$$

**Example 7:** Simplify:  $\frac{\sin(x)}{\cos^2(x)-1} = \boxed{-\csc(x)}$

Recall:

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \cos^2 x - 1 = -\sin^2 x$$

$$\Rightarrow \frac{\sin x}{\cos^2 x - 1} = \frac{\sin x}{-\sin^2 x}$$

$$= - \frac{\cancel{\sin x}}{\cancel{\sin x} \cdot \sin x}$$

$$= -\frac{1}{\sin x} = \boxed{-\csc x}$$

$$= \csc(-x) \leftarrow \text{a possible answer.}$$

**Example 8:** Simplify:  $\frac{\sec^2(\theta)}{\tan(\theta) + \cot(\theta)} = \boxed{\tan \theta}$

$$\frac{\sec^2 \theta}{\tan \theta + \cot \theta} = \frac{\frac{1}{\cos^2 \theta}}{\frac{\sin \theta}{\cos \theta} \cdot \frac{\sin \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{\cos \theta}}$$

$$= \frac{\frac{1}{\cos^2 \theta}}{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}} = 1$$

$$= \frac{\frac{1}{\cos^2 \theta}}{\frac{1}{\sin \theta \cos \theta}}$$

$$= \frac{1}{\cos \theta \cdot \cancel{\cos \theta}} \cdot \frac{\sin \theta \cdot \cancel{\cos \theta}}{1} = \frac{\sin \theta}{\cos \theta} = \boxed{\tan \theta}$$

Or:

$$\frac{\sec^2 \theta}{\tan \theta + \cot \theta} = \frac{\sec^2 \theta}{\frac{\tan \theta}{\tan \theta} \tan \theta + \frac{1}{\tan \theta}} = \frac{\sec^2 \theta}{\frac{\tan^2 \theta + 1}{\tan \theta}} = \sec^2 \theta$$

$$= \frac{\sec^2 \theta}{\frac{\sec^2 \theta}{\tan \theta}} = \frac{\cancel{\sec^2 \theta}}{1} \cdot \frac{\tan \theta}{\cancel{\sec^2 \theta}} = \boxed{\tan \theta}$$



**Example 9:** Simplify:  $\frac{1 - \sin^2(\theta)}{\cos^2(\theta) - 1} = \boxed{-\cot^2 \theta}$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\Rightarrow 1 - \sin^2 \theta = \cos^2 \theta$$

and

$$\cos^2 \theta - 1 = -\sin^2 \theta$$

$$\Rightarrow \frac{1 - \sin^2 \theta}{\cos^2 \theta - 1} = \frac{\cos^2 \theta}{-\sin^2 \theta}$$

$$= - \left( \frac{\cos \theta}{\sin \theta} \right)^2$$

$$= \boxed{-\cot^2 \theta.}$$

## exercise

**Example 10:** Simplify:  $\frac{\cot(x)}{\csc(x)-1} + \frac{\cot(x)}{\csc(x)+1} = \boxed{2 \sec x}$

$$\frac{\cot(x)}{\csc(x)-1} + \frac{\cot(x)}{\csc(x)+1} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x} - 1 \cdot \frac{\sin x}{\sin x}} + \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x} + 1 \cdot \frac{\sin x}{\sin x}}$$

$$= \frac{\frac{\cos x}{\cancel{\sin x}}}{\frac{1-\sin x}{\cancel{\sin x}}} + \frac{\frac{\cos x}{\cancel{\sin x}}}{\frac{1+\sin x}{\cancel{\sin x}}}$$

$$= \frac{\cos x}{1-\sin x} + \frac{\cos x}{1+\sin x}$$

$$= \frac{(\cos x)(1+\sin x)}{(1-\sin x)(1+\sin x)} + \frac{(\cos x)(1-\sin x)}{(1+\sin x)(1-\sin x)}$$

$$= \frac{\cos x + \cancel{\cos x \sin x} + \cos x - \cancel{\cos x \sin x}}{1 - \sin^2 x} = \cos^2 x$$

$$= \frac{2 \cancel{\cos x}}{\cos x \cdot \cancel{\cos x}} = \boxed{2 \sec(x)}$$

You can also use the identities to help you solve problems like this one. (Note: you can also use a triangle to help you work this problem.)

**Example 11:** If  $\cot(\theta) = \frac{5}{12}$ , where  $\pi < \theta < \frac{3\pi}{2}$ , find the exact values of  $\tan(\theta)$  and  $\sec(\theta)$ .

Use Identities:

$$\bullet \quad \boxed{\tan \theta = \frac{1}{\cot \theta} = \frac{1}{5/12} = \frac{12}{5}}$$

$$\bullet \quad \boxed{1 + \tan^2 \theta = \sec^2 \theta}$$

$$\sec^2 \theta = 1 + \left(\frac{12}{5}\right)^2$$

$$\sec^2 \theta = \frac{169}{25},$$

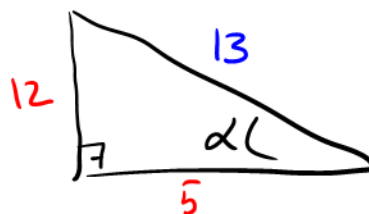
$$\Rightarrow \sec \theta = -\sqrt{\frac{169}{25}} = -\frac{13}{5}$$

$$\Rightarrow \boxed{\tan \theta = \frac{12}{5}, \sec \theta = -\frac{13}{5}}$$

Use triangle:

$$\cot \theta = \frac{5}{12}, \quad \theta \leftrightarrow \text{III}$$

reference  $\alpha$



$$\cot \alpha = \frac{\text{Adj}}{\text{Opp}}$$

$$\Rightarrow \tan \alpha = \frac{\text{Opp}}{\text{Adj}} = \frac{12}{5}$$

$$\sec \alpha = \frac{\text{Hyp}}{\text{Adj}} = \frac{13}{5}$$

$$\Rightarrow \boxed{\tan \theta = \frac{12}{5}, \sec \theta = -\frac{13}{5}}$$

At times, you may be asked to verify identities. To do this, you'll use the identities and algebraic operations to show that the left-hand side of the problem equals the right-hand side of the problem.

$$\text{LHS} = \text{RHS}$$

Here are some pointers for helping you verify identities:

1. Remember that your task is to show that the two sides of the equation are equal. You may not assume that they are equal. *You prove LHS = RHS.*
2. *Very important* Choose one side of the problem to work with and leave the other one alone. You'll use identities and algebra to convert one side so that it is identical to the side you left alone. You'll work with the "ugly" or more complicated side.
3. It is often helpful to convert all trig functions into sine and cosine. This is usually very helpful! (Unless it makes things worse!!)
4. Find common denominators, if appropriate.
5. Don't try to do too much in one step. Take it one step at a time!
6. If working with one side doesn't get you anywhere, try working with the other side instead.

**Example 12:** Prove the identity:

$$\frac{\sin x \cos x}{1 - \cos^2 x} = \cot x$$

$$\text{LHS} = \frac{\sin x \cdot \cos x}{1 - \cos^2 x}, \quad \text{RHS} = \cot x, \quad \text{Show LHS} = \text{RHS}.$$

Begin with one side:

$$\text{LHS} = \frac{\sin x \cdot \cos x}{1 - \cos^2 x} = \frac{\sin x \cos x}{\sin^2 x} = \frac{\cancel{\sin x} \cos x}{\cancel{\sin x} \sin x}$$

$$= \frac{\cos x}{\sin x} = \cot x = \text{RHS}.$$

Done

left hand
right hand

**Example 13:** Prove the identity:  $\cos^2(x) - \sin^2(x) = \frac{1 - \tan^2(x)}{1 + \tan^2(x)}$

LHS
RHS

to prove identities:

- Choose one side  
(the one that is more challenging)

RHS =  $\frac{1 - \tan^2 x}{1 + \tan^2 x}$  = do all possible operations until you convert to the other side.

$$= \frac{1 - \tan^2 x}{\sec^2 x} = \frac{1 - \tan^2 x}{\frac{1}{\cos^2 x}}$$

$$= (1 - \tan^2 x) \cdot \frac{\cos^2 x}{1} = \cos^2 x - \frac{\cancel{\cos^2 x}}{1} \cdot \frac{\sin^2 x}{\cancel{\cos^2 x}}$$

$$= \cos^2 x - \sin^2 x = \text{LHS}$$

**Example 13:** Prove the identity:  $\underbrace{\cos^2(x) - \sin^2(x)}_{\text{LHS}} = \underbrace{\frac{1 - \tan^2(x)}{1 + \tan^2(x)}}_{\text{RHS}}$

$$\text{RHS} = \frac{1 - \tan^2 x}{\underbrace{1 + \tan^2 x}_{= \sec^2 x}} = \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\sec^2 x} = \frac{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

$$= \frac{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} = \frac{\cos^2 x - \sin^2 x}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos^2 x}}{1}$$

$$= \cos^2 x - \sin^2 x = \text{LHS.}$$

**DONE**