

Popper #25

(1)

$$\frac{1 - \sin^2 \theta}{\sin^2 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta} = \cot^2 \theta$$

(A)

- A. $\cot^2 \theta$ B. 1 C. $\csc^2 \theta$ D. $\tan^2 \theta$ E. none

(2)

Which of the followings are true?

I. $\sin^2 \theta + \cos^2 \theta = 1$ ✓

II. $\tan^2 \theta + 1 = \sec^2 \theta$ ✓

III. $\cot^2 \theta + 1 = \csc^2 \theta$ ✓

IV. $\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} = 1$ ✓

↳ $\cos^2 \theta + \sin^2 \theta = 1$

- A. I, II, III

- B. All are true

- C. I, II

(3)

A

(4)

B

(5)

C

Warm - Up :

We'll learn $\sin(A+B)$ or $\sin(A-B)$

$\cos(A+B)$ or $\cos(A-B)$

$\tan(A+B)$ or $\tan(A-B)$, and so on ...

Never forget:

$$\sin(A+B) \neq \sin(A) + \sin(B)$$

Ex.

$$\begin{aligned} & \sin(30^\circ + 60^\circ) \\ &= \sin(90^\circ) \\ &= 1 \end{aligned} \quad \left. \begin{array}{l} \sin(30^\circ) + \sin(60^\circ) \\ = \frac{1}{2} + \frac{\sqrt{3}}{2} \\ = \frac{1+\sqrt{3}}{2} \end{array} \right\}$$

$$\Rightarrow \sin(30^\circ + 60^\circ) \neq \sin(30^\circ) + \sin(60^\circ)$$

You can find similar examples to show
that trigonometric functions are not linear.

Section 6.1 - Sum and Difference Formulas

Note: $\sin(A + B) \neq \sin(A) + \sin(B)$
 $\cos(A + B) \neq \cos(A) + \cos(B)$

Sum and Difference Formulas for Sine, Cosine and Tangent

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Very
important

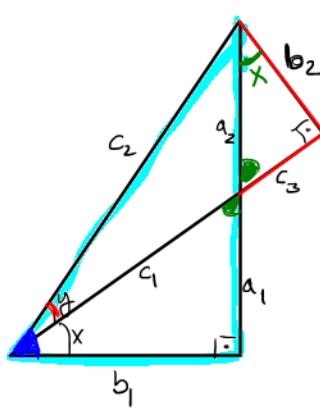
Why do we need them?

Well, we can calculate a lot of angles without using calculator.

$$\begin{aligned}\sin(75^\circ) &= \sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \\&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4} \leftarrow \text{exact value.}\end{aligned}$$

How to prove one of the identities, say:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$



$$\boxed{\sin(x+y) = \frac{a_1 + a_2}{c_2}}$$

$$\sin x = \frac{a_1}{c_1} = \frac{c_3}{a_2}$$

$$\cos x = \frac{b_1}{c_1} = \frac{b_2}{a_2}$$

$$\sin y = \frac{b_2}{c_2}$$

$$\cos y = \frac{a_1 + c_3}{c_2}$$

Cross-product

$$c_1 \cdot c_3 = a_1 a_2$$

$$\sin x \cos y + \cos x \sin y = \frac{c_3}{a_2} \cdot \frac{(c_1 + c_3)}{c_2} + \frac{b_2}{a_2} \cdot \frac{b_2}{c_2}$$

$$= \underbrace{c_3 \cdot c_1}_{a_1 a_2} + \underbrace{\frac{c_3^2}{c_3 + b_2}}_{c_3^2 + b_2^2} = a_2^2$$

$$a_2^2$$

$$= \frac{a_1 a_2 + a_2^2}{a_2 \cdot c_2} = \frac{\cancel{a_2}(a_1 + a_2)}{\cancel{a_2} \cdot c_2} = \frac{a_1 + a_2}{c_2}$$

$$= \sin(x+y)$$

It does work !!

To be continued on Wednesday, 04/04

Example 1: Simplify each:

a. $\cos(x+60^\circ) =$ \rightarrow We'll use $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \cos x \cos 60^\circ - \sin x \sin 60^\circ$$

$$= \cos x \cdot \frac{1}{2} - \sin x \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{\cos x - \sqrt{3} \sin x}{2}}$$

b. $\sin\left(x + \frac{\pi}{4}\right) - \sin\left(x - \frac{\pi}{4}\right)$ \rightarrow We'll use

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

||

$$\boxed{\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}} - \boxed{\sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4}}$$

$$= \cos x \sin \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} = 2 \cos x \sin \frac{\pi}{4}$$

$$= \cancel{2} \cos x \cdot \frac{\sqrt{2}}{\cancel{2}} = \boxed{\sqrt{2} \cos x}$$

Example 2: Given that $\tan(x) = 5$, evaluate $\tan\left(x + \frac{3\pi}{4}\right)$.

We'll use $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\begin{aligned} \Rightarrow \tan\left(x + \frac{3\pi}{4}\right) &= \frac{\tan x + \tan\left(\frac{3\pi}{4}\right)}{1 - \tan x \cdot \tan\left(\frac{3\pi}{4}\right)} \\ &= \frac{5 + (-1)}{1 - 5 \cdot (-1)} = \frac{4}{6} = \boxed{\frac{2}{3}} \end{aligned}$$

Example 3: Simplify each.

We'll use $\sin(x-y) = \sin x \cos y - \cos x \sin y$
backwards

a. $\sin 10^\circ \cos 55^\circ - \sin 55^\circ \cos 10^\circ$

$$= \sin 10^\circ \cos 55^\circ - \cos 10^\circ \cdot \sin 55^\circ$$

$$= \sin(10^\circ - 55^\circ) = \sin(-45^\circ) = -\sin 45^\circ =$$

$$\boxed{-\frac{\sqrt{2}}{2}}$$

b. $\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\sin\left(\frac{7\pi}{12}\right)$

We'll use $\cos(x-y) = \cos x \cos y + \sin x \sin y$
backwards

$$\hookrightarrow = \cos\left(\frac{\pi}{12} - \frac{7\pi}{12}\right)$$

$$= \cos\left(-\frac{6\pi}{12}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

c. $\frac{\tan 40^\circ + \tan 5^\circ}{1 - \tan 40^\circ \tan 5^\circ} = \tan(40^\circ + 5^\circ)$

$$= \tan 45^\circ = 1$$

Use $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
backwards

d. $\frac{\tan 80^\circ - \tan 15^\circ}{1 + \tan 80^\circ \tan 15^\circ} = \tan(80^\circ - 15^\circ)$

$$= \tan(65^\circ)$$

Use $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
backwards

Example 4: Find the exact value of each. (Hint: use sum/difference formulas)

a. $\sin 15^\circ$

$$\begin{aligned} & \parallel \\ & \sin(45^\circ - 30^\circ) \\ & \parallel \end{aligned}$$

$$\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$\begin{aligned} & \parallel \\ & \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

$30^\circ, 45^\circ, 60^\circ$
 $\underbrace{}$
difference = 15°

b. $\cos\left(\frac{7\pi}{12}\right) \rightarrow \text{think } \frac{7\pi}{12} = 105^\circ$

||

$$\underbrace{45^\circ + 60^\circ}$$

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

||

$$\cos \frac{\pi}{4} \cdot \cos \frac{\pi}{3} - \sin \frac{\pi}{4} \sin \frac{\pi}{3}$$

||

$$\begin{aligned} & \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4} \end{aligned}$$

30°	45°	60°	90°
$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
		<u>check</u>	

$$\rightarrow \text{the sum } \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$$

$$\begin{array}{ccccccc}
 30^\circ & 45^\circ & 60^\circ & 90^\circ & 120^\circ & 135^\circ & 150^\circ & 180^\circ \\
 \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} & \frac{\pi}{2} & \frac{2\pi}{3} & \frac{3\pi}{4} & \frac{5\pi}{6} & \pi
 \end{array}$$

their sum

c. $\tan\left(\frac{5\pi}{12}\right) \rightarrow \frac{5\pi}{12} = 75^\circ = 30^\circ + 45^\circ , \quad \frac{4\pi}{6} + \frac{\pi}{4} = \frac{10\pi}{24} = \frac{5\pi}{12}$

||

$$\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

||

$$\frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right) \cdot \tan\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{3}}{3} + 1 \cdot \frac{3}{3}}{\frac{3}{3} \cdot 1 - \frac{\sqrt{3}}{3} \cdot 1} = \frac{\frac{\sqrt{3}+3}{3}}{\frac{3-\sqrt{3}}{3}} = \frac{\sqrt{3}+3}{3-\sqrt{3}}$$

Very Important $\rightarrow = \frac{(\sqrt{3}+3)}{3-\sqrt{3}} \cdot \frac{(3+\sqrt{3})}{(3+\sqrt{3})} \stackrel{\text{FOIL}}{=} \frac{3\sqrt{3} + (\sqrt{3})^2 + 9 + 3\sqrt{3}}{3^2 - (\sqrt{3})^2}$

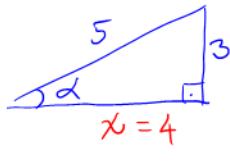
$$= \frac{6\sqrt{3} + 12}{9 - 3} = \frac{6\sqrt{3} + 12}{6} = \frac{6(\sqrt{3} + 2)}{6}$$

$$= \boxed{\sqrt{3} + 2}$$

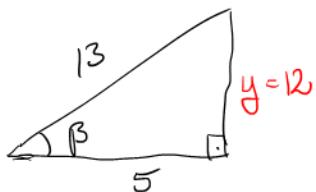
both angles in Quadrant I

Example 5: Suppose that $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{5}{13}$ where $0 < \alpha < \beta < \frac{\pi}{2}$. Find each of these:

$$\sin \alpha = \frac{3}{5} \Rightarrow \cos \alpha = \frac{4}{5}$$



$$\cos \beta = \frac{5}{13} \Rightarrow \sin \beta = \frac{12}{13}$$



a. $\sin(\alpha + \beta)$

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$$\sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$= \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13}$$

$$= \frac{15}{65} + \frac{48}{65} = \boxed{\frac{63}{65}}$$

b. $\cos(\alpha - \beta)$

||

$$\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

$$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13}$$

$$= \frac{20}{65} + \frac{36}{65} = \boxed{\frac{56}{65}}$$

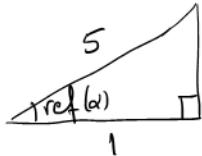
Think: $\begin{cases} \pi < \alpha < 2\pi \\ \cos \alpha = \frac{1}{5} \text{ positive} \end{cases}$
 $\Rightarrow \alpha$ in Quadrant IV

$\begin{cases} \pi < \beta < 2\pi \\ \tan \beta = -\frac{7}{6} \text{ negative} \end{cases}$
 $\Rightarrow \beta$ in Quadrant IV

Exercise

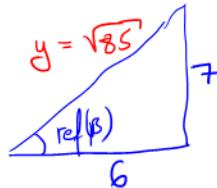
Example 6: Suppose $\cos \alpha = \frac{1}{5}$ and $\tan \beta = -\frac{7}{6}$ where $\pi < \alpha, \beta < 2\pi$. Find

$\Rightarrow \sin \alpha = \text{negative}$



$$\Rightarrow \sin \alpha = -\frac{2\sqrt{6}}{5}$$

$\Rightarrow \sin \beta = \text{negative}, \cos \beta = \text{positive}$



$$\sin \beta = -\frac{7}{\sqrt{85}} = -\frac{7\sqrt{85}}{85}$$

$$\cos \beta = \frac{6}{\sqrt{85}} = \frac{6\sqrt{85}}{85}$$

$$y = \sqrt{6^2 + 7^2} = \sqrt{85}$$

a. $\cos(\alpha + \beta)$

||

$$\cos \alpha \cdot \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{1}{5} \cdot \frac{6\sqrt{85}}{85} - \left(-\frac{2\sqrt{6}}{5} \right) \left(-\frac{7\sqrt{85}}{85} \right)$$

$$= \frac{6\sqrt{85}}{5 \cdot 85} - \frac{14\sqrt{6} \cdot \sqrt{85}}{5 \cdot 85} = \frac{6\sqrt{85} - 14\sqrt{510}}{425}$$

b. $\tan(\alpha + \beta)$

$$= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\text{Note: } \tan \alpha = -\frac{2\sqrt{6}}{1} = -2\sqrt{6}$$

$$\tan \beta = -\frac{7}{6}$$

$$= \frac{\frac{6}{6} \cdot \frac{-2\sqrt{6}}{1} - \frac{7}{6}}{\frac{6}{6} \cdot 1 - (-2\sqrt{6}) \left(-\frac{7}{6} \right)} =$$

$$= \frac{\frac{-12\sqrt{6} - 7}{6}}{\frac{6}{6} - \frac{14\sqrt{6}}{6}} = \frac{-12\sqrt{6} - 7}{6 - 14\sqrt{6}}$$

$$= \frac{-12\sqrt{6} - 7}{6} \cdot \frac{6}{6 - 14\sqrt{6}}$$

$$= \frac{(-1) \frac{-12\sqrt{6} - 7}{6}}{(-1) \frac{6 - 14\sqrt{6}}{6}} = \boxed{\frac{12\sqrt{6} + 7}{14\sqrt{6} - 6}}$$

Popper 26

1 $\sin 15^\circ \cos 75^\circ + \cos 15^\circ \sin 75^\circ = \sin(15^\circ + 75^\circ) = \sin 90^\circ = 1$

- A. 0 B. 1 C. $\frac{\sqrt{3}}{2}$ D. $\frac{1}{2}$ E. none

Be careful

2 $\boxed{\sin 55^\circ} \boxed{\cos 10^\circ} - \boxed{\cos 10^\circ} \boxed{\sin 55^\circ} = 0$
they are same

- A. 0 B. $\frac{\sqrt{2}}{2}$ C. $-\frac{\sqrt{2}}{2}$ D. 1 E. none

3
$$\frac{\tan(\underline{6x})^A + \tan(\underline{\pi})^B}{1 - \tan(\underline{6x}) \cdot \tan\underline{\frac{\pi}{2}}} = \tan(6x + \cancel{\pi}) = \tan(6x)$$

- A. $\tan(6x)$ B. $\tan(3x)$ C. 0 D. none

4 $\boxed{\sin\left(\frac{\pi}{2} - x\right) = \cos x}$

$$\sin\left(\frac{\pi}{2} - x\right) = \cancel{\sin \frac{\pi}{2}} \cdot \cos x - \cos \cancel{\frac{\pi}{2}} \cdot \sin x = 0$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

- A. $\sin x$ B. $\cos x$ C. $-\sin x$ D. $-\cos x$

5 $\boxed{\cos\left(\frac{\pi}{2} - x\right) = \sin x}$ Similar

- A. $\sin x$ B. $\cos x$ C. $-\sin x$ D. $-\cos x$