Popper # 27

1. $$\frac{2-\cot^2\theta}{\underbrace{1+\cot^2\theta}_{\csc^2\theta=\frac{1}{\sin^2\theta}}} + 3\cos^2\theta = \frac{2\cdot\frac{\sin^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}} + 3\cos^2\theta$$

$$= \frac{2\sin^2\theta-\cos^2\theta}{\cancel{\sin^2\theta}}\cdot\frac{\cancel{\sin^2\theta}}{1} + 3\cos^2\theta$$

$$= 2\sin^2\theta + 2\cos^2\theta = 2$$

A. 1 B. 2 C. $2\cos^2\theta$ D. $2\sin^2\theta$

2. $$\frac{1-\tan^2 x}{\underbrace{1+\tan^2 x}_{\sec^2 x=\frac{1}{\cos^2 x}}} + 2\sin^2 x = \frac{1\cdot\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}} + 2\sin^2 x$$

$$= \cos^2 x - \sin^2 x + 2\sin^2 x$$

$$= \cos^2 x + \sin^2 x = 1$$

A. $2\sin^2 x$ B. 1 C. $2\cos^2 x$ D. 2

3. A

4. B

Section 6.2 – Double and Half Angle Formulas

Now suppose we are interested in finding $\sin(2A)$. We can use the sum formula for sine to develop this identity:

Recall $\sin(A+B) = \sin A \cos B + \cos A \sin B$, $A = B$

$$\sin(2A) = \sin(A+A)$$
$$= \sin A \cos A + \sin A \cos A$$
$$\boxed{\sin 2A = 2\sin A \cos A}$$

Similarly, we can develop a formula for $\cos(2A)$:

Recall $\cos(A+B) = \cos A \cos B - \sin A \sin B$, $A = B$

$$\cos(2A) = \cos(A+A)$$
$$= \cos A \cos A - \sin A \sin A$$
$$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$$

We can restate this formula in terms of sine only or in terms of cosine only by using the Pythagorean theorem and making a substitution. So we have:

$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$\cos(2A) = 1 - 2\sin^2 A$$
$$\cos(2A) = 2\cos^2 A - 1$$

$\Longleftarrow$ Combine

If $\cos^2 A + \sin^2 A = 1$, then

$\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$

We can also develop a formula for $\tan(2A)$:

Recall $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, $A = B$

$$\tan(2A) = \tan(A+A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$
$$\boxed{\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}}$$

These three formulas are called the **double angle formulas for sine, cosine and tangent**.

Besides these formulas, we also have the so-called **half-angle formulas for sine, cosine and tangent**, which are derived by using the double angle formulas for sine, cosine and tangent, respectively.

Double – Angle Formulas

$$\sin(2A) = 2\sin A\cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

Learn to apply correctly!!!

Half – Angle Formulas , call $\frac{A}{2} = x$, then $A = 2\cdot\frac{A}{2} = 2x$.

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

Rewrite $\Longrightarrow$

$$\sin(x) = \pm\sqrt{\frac{1-\cos(2x)}{2}}$$

$$\cos(x) = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

$$\tan(x) = \frac{\sin(2x)}{1+\cos(2x)} = \frac{1-\cos(2x)}{\sin(2x)}$$

Note: In the half-angle formulas the ± symbol is intended to mean either positive or negative but not both, and the sign before the radical is determined by the quadrant in which the angle $\frac{A}{2}$ terminates.

Example 1: Suppose that $\cos\alpha = -\frac{4}{7}$ and $\frac{\pi}{2} < \alpha < \pi$. Find

Quadrant II, $\sin\alpha$ positive

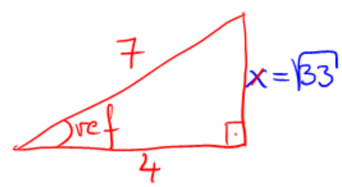
a. $\cos(2\alpha)$ $= 2\cos^2\alpha - 1$

$$= 2\left(-\frac{4}{7}\right)^2 - 1 = 2\cdot\frac{16}{49} - 1 = \frac{32}{49} - \frac{49}{49} = \boxed{\frac{-17}{49}}$$

b. $\sin(2\alpha)$ $= 2\sin\alpha\cdot\cos\alpha$

$$= 2\left(\frac{\sqrt{33}}{7}\right)\cdot\left(\frac{-4}{7}\right)$$

$$= \boxed{\frac{-8\sqrt{33}}{49}}$$



$\sin\alpha = \frac{\sqrt{33}}{7}$

$\tan\alpha = -\frac{\sqrt{33}}{4}$

c. $\tan(2\alpha)$

$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)}$

$$= \frac{-8\sqrt{33}/49}{-17/49} = \frac{8\sqrt{33}}{17}$$

$$= \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{\sqrt{33}}{4}\right)}{1-\left(-\frac{\sqrt{33}}{4}\right)^2} = \frac{-\frac{\sqrt{33}}{2}}{\frac{16}{16}\cdot 1 - \frac{33}{16}} = \frac{-\sqrt{33}/2}{-17/16}$$

$$= \frac{\sqrt{33}}{\cancel{2}}\cdot\frac{\cancel{16}^{\,8}}{17} = \boxed{\frac{8\sqrt{33}}{17}}$$

Example 2: Simplify each:

DOUBLE ANGLE FORMULAS — BACKWARDS

a. $2\sin 45°\cos 45°$

$$= \sin(2\cdot 45°) = \sin(90°) = \boxed{1}$$

b. $\cos^2\frac{\pi}{9}-\sin^2\frac{\pi}{9}$ $= \cos\left(2\times\frac{\pi}{9}\right) = \cos\left(\frac{2\pi}{9}\right)$

c. $\frac{2\tan 15^\circ}{1-\tan^2 15^\circ}$ $= \tan(2\times 15^\circ)$

$= \tan(30^\circ) = \frac{\sqrt{3}}{3}$

d. $1-2\sin^2(6A)$ $= \cos(2 \cdot 6A) = \cos(12A)$

this is your angle, so you double it

Now we'll look at the kinds of problems we can solve using half-angle formulas.

Recall:

- $\cos(2A) = 2\cos^2(A) - 1$ ← Solve for $\cos(A)$

$+1 \qquad\qquad +1$

$\cos(2A) + 1 = 2\cos^2 A$

or $2\cos^2(A) = \cos(2A) + 1$ ← Divide by 2

$$\cos^2(A) = \frac{\cos(2A)+1}{2} \leftarrow \text{Take } \sqrt{\cdot}$$

$$\boxed{\cos(A) = \pm\sqrt{\frac{1+\cos(2A)}{2}}}$$

$$\cos(2A) = 1 - 2\sin^2(A) \leftarrow \text{Solve for } \sin(A)$$

$$+2\sin^2(A) \qquad +2\sin^2(A)$$

$$-\cancel{\cos(2A)} \qquad -\cos(2A)$$

$$2\sin^2(A) = 1 - \cos(2A) \leftarrow \text{Divide by } 2$$

$$\sin^2(A) = \frac{1-\cos(2A)}{2} \leftarrow \text{Take } \sqrt{\cdot}$$

$$\boxed{\sin(A) = \pm\sqrt{\frac{1-\cos(2A)}{2}}}$$

$$\tan(A) = \frac{\sin(A)}{\cos(A)} = \frac{\sqrt{\frac{1-\cos(2A)}{\cancel{2}}}}{\sqrt{\frac{1+\cos(2A)}{\cancel{2}}}}$$

Suppose A is in Quadrant I.

$$= \sqrt{\frac{1-\cos(2A)}{1+\cos(2A)} \cdot \frac{(1-\cos(2A))}{(1-\cos(2A))}} = \sqrt{\frac{(1-\cos(2A))^2}{1-\cos^2(2A)}}$$

$$= \frac{\sqrt{(1-\cos(2A))^2}}{\sqrt{\sin^2(2A)}} = \boxed{\frac{1-\cos(2A)}{\sin(2A)}}$$

similarly for the other form.

$\sin(x) = \pm\sqrt{\dfrac{1-\cos(2x)}{2}}$, $\cos(x) = \pm\sqrt{\dfrac{1+\cos(2x)}{2}}$, $\tan(x) = \dfrac{\sin(2x)}{1+\cos(2x)} = \dfrac{1-\cos(2x)}{\sin(2x)}$

Example 3: Use a half-angle formula to find the exact value of each.

a. $\sin 15°$ = positive

$$= \sqrt{\frac{1-\cos(2\cdot 15°)}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

Think

$\frac{5\pi}{8} = \frac{5\cancel{\pi}}{\cancel{8}_2} \cdot \frac{\cancel{180}^{45}}{\cancel{\pi}} = \frac{225}{2}$

b. $\cos\left(\dfrac{5\pi}{8}\right) = \cos\left(\overset{\text{quadrant II}}{\dfrac{225°}{2}}\right)$ = negative

$$= -\sqrt{\frac{1+\cos\left(2\cdot\frac{225°}{2}\right)}{2}} = -\sqrt{\frac{1+\cos(225°)}{2}}$$

$$= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}} = \boxed{-\frac{\sqrt{2-\sqrt{2}}}{2}}$$

c. $\tan\left(\overset{\text{II}}{\dfrac{7\pi}{12}}\right) = \tan(105°)$

Think

$\frac{7\pi}{12} = \frac{7\cancel{\pi}}{\cancel{12}} \cdot \frac{\cancel{180}^{15}}{\cancel{\pi}} = 105°$

$\tan x = \dfrac{\sin(2x)}{1+\cos(2x)}$

$$= \frac{\sin(2\cdot 105°)}{1+\cos(2\cdot 105°)} = \frac{\sin(210°)}{1+\cos(210°)} = \frac{-1/2}{1-\sqrt{3}/2}$$

$$= \frac{-\frac{1}{\cancel{2}}}{\frac{2-\sqrt{3}}{\cancel{2}}} = -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{2-\sqrt{3}} = \frac{-1}{2-\sqrt{3}}\cdot\frac{(2+\sqrt{3})}{(2+\sqrt{3})}$$

$$= \frac{-(2+\sqrt{3})}{2^2-(\sqrt{3})^2} = \frac{-(2+\sqrt{3})}{4-3} = \boxed{-2-\sqrt{3}}$$

(c) $\tan\left(\frac{7\pi}{12}\right) = \tan(105^\circ)$

$\tan x = \frac{1-\cos(2x)}{\sin(2x)}$

$$= \frac{1-\cos(2 * 105^\circ)}{\sin(2 * 105^\circ)}$$

$$= \frac{1-\overset{-\sqrt{3}/2}{\cos(210^\circ)}}{\sin(210^\circ)} = \frac{\frac{2}{2}\cdot 1 + \frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\frac{2+\sqrt{3}}{2}}{-1/2}$$

$$= -(2+\sqrt{3}) = \boxed{-2-\sqrt{3}}$$

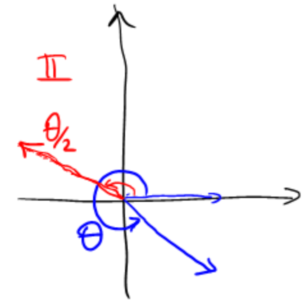
To be continued on Monday, 04/11

Quadrant IV

Example 4: Answer these questions for $\cos\theta = \frac{4}{9}, \frac{3\pi}{2} < \theta < 2\pi$.

Divide by 2.

$$\frac{\frac{3\pi}{2}}{2} < \frac{\theta}{2} < \frac{2\pi}{2}$$



a. In which quadrant does the terminal side of the angle lie?

b. Complete the following: $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$

Half of the angle θ should be in Q.II.

c. In which quadrant does the terminal side of $\frac{\theta}{2}$ lie? Quadrant II.

d. Determine the sign of $\sin\left(\frac{\theta}{2}\right)$ = positive

e. Determine the sign of $\cos\left(\frac{\theta}{2}\right)$ = negative

f. Find the exact value of $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\frac{9}{9}\cdot 1-\frac{4}{9}}{2}} = \sqrt{\frac{\frac{5}{9}}{2}} = \sqrt{\frac{5}{18}}$

$$= \frac{\sqrt{5}}{\sqrt{18}} = \frac{\sqrt{5}}{\sqrt{9}\cdot\sqrt{2}} = \frac{\sqrt{5}}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{10}}{6}}$$

g. Find the exact value of $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{\frac{9}{9}\cdot 1+\frac{4}{9}}{2}} = -\sqrt{\frac{\frac{13}{9}}{2}} = -\sqrt{\frac{13}{18}}$

$$= \frac{-\sqrt{13}}{\sqrt{18}} = \frac{-\sqrt{13}}{\sqrt{9}\sqrt{2}} = \frac{-\sqrt{13}}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \boxed{-\frac{\sqrt{26}}{6}}$$

$81-16 = 65$

h. Find the exact value of $\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{\sqrt{10}/\cancel{6}}{-\sqrt{26}/\cancel{6}} = -\sqrt{\frac{10}{26}} = -\sqrt{\frac{5}{13}} = \boxed{-\frac{\sqrt{65}}{13}}$

or

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{4}{9}}{-\frac{\sqrt{65}}{9}} = \frac{5/\cancel{9}}{-\sqrt{65}/\cancel{9}} = \frac{-5}{\sqrt{65}} = \frac{-\cancel{5}\sqrt{65}}{\cancel{65}_{13}} = \boxed{\frac{-\sqrt{65}}{13}}$$

match

2 find

Quadrant IV

$\sin\theta = (-)\sqrt{1-\left(\frac{4}{9}\right)^2} = -\frac{\sqrt{65}}{9}$