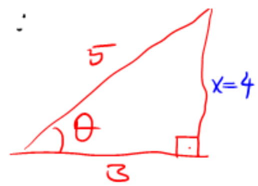
Popper # 28

$1 + \tan^2\theta = \sec^2\theta$

① $(\sec\theta - 1)(\sec\theta + 1) = \sec^2\theta - 1 = \tan^2\theta$

A. $\sec^2\theta$ B. $\tan^2\theta$ C. $\sin^2\theta$ D. $\cos^2\theta$ E. $\cot^2\theta$

→ Given $\cos\theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$, find:



$\cos\theta = \frac{3}{5}$

$\sin\theta = \frac{4}{5}$

② $\sin(2\theta) = 2\sin\theta \cdot \cos\theta = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}$

A. $\frac{6}{25}$ B. $\frac{8}{25}$ C. $\frac{24}{25}$ D. $\frac{4}{25}$ E. none

③ $\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{3}{5}\right)^2 - 1 = 2 \cdot \frac{9}{25} - \frac{25}{25} = \frac{-7}{25}$

A. $\frac{7}{25}$ B. $-\frac{7}{25}$ C. $\frac{18}{25}$ D. $\frac{-18}{25}$ E. none

④ $\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{24/25}{-7/25} = -24/7$

A. $\frac{24}{7}$ B. $\frac{-24}{7}$ C. $\frac{4}{3}$ D. $\frac{16}{9}$ E. none.

⑤ $\tan(\theta/2) = \frac{\sin\theta}{1 + \cos\theta} = \frac{4/5}{1 + 3/5} = \frac{4/5}{8/5} = \frac{4}{8} = \frac{1}{2}$

A. $2/5$ B. $2/3$ C. $1/2$ D. $5/3$ E. none

How to solve a Linear Equation: $2x + 3 = 9$ ← Solve!

$\underset{-3}{}\quad\underset{-3}{}$

"Goal is to leave x alone"

$\dfrac{2x}{2} = \dfrac{6}{2} \implies \boxed{x = 3}$

Section 6.3 - Solving Trigonometric Equations

Next, we'll use all of the tools we've covered in our study of trigonometry to solve some equations. An equation that involves a trigonometric function is called a trigonometric equation. Since trigonometric functions are periodic, there may be infinitely solutions to some trigonometric equations.

- Trigonometric Equation ← Unit Circle

Let's say we want to solve the equation: $\sin(x) = \dfrac{1}{2}$

Ask yourself: Which angle(s) have sine $= \frac{1}{2}$?

The first angles that come to mind are: $\boxed{x = \dfrac{\pi}{6} \text{ and } x = \dfrac{5\pi}{6}}$. ← Answer.

in one period

$x = 30°$ or $150°$

(convert in radians)

Remember that the period of the sine function is 2π; sine function repeats itself after each rotation.

The solutions of unit circle repeat themselves in every periodic rotation.

Therefore, the solutions of the equation are: $x = \underbrace{\dfrac{\pi}{6}}_{\text{unit circle}} + \underbrace{2k\pi}_{2\pi \cdot k}$, $x = \underbrace{\dfrac{5\pi}{6}}_{\text{unit circle}} + \underbrace{2k\pi}_{2\pi \cdot k}$, where k is any integer.

Recall: For sine and cosine functions, the period is 2π. For tangent and cotangent functions, the period is π.

DO NOT FORGET:

General (ALL) Solutions = $\underbrace{\text{Special Solutions}}_{\text{unit circle}}$ + Period $\cdot$ k

To be continued on Wednesday, 04/13

one rotation

Example 1: a) Solve the equation in the interval $[0, 2\pi)$: $2\cos x = -1$

$2\cos x = -1$

Q.II QIII

$\cos x = -\frac{1}{2} \Longrightarrow x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

period $= 2\pi$

b) Find all solutions to the equation: $2\cos x = -1$

From part (a), $\cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

all $\boxed{x = \frac{2\pi}{3} + 2\pi \cdot k \quad \text{or} \quad x = \frac{4\pi}{3} + 2\pi \cdot k,}$ k integer

one period

Example 2: a) Solve the equation in the interval $[0, \pi)$: $\tan x = -1$

$\tan x = -1$

From 0 to π,

only $\boxed{x = \frac{3\pi}{4}}$ in Quadrant II, gives $\tan\left(\frac{3\pi}{4}\right) = -1$

b) Find all solutions to the equation: $\tan x = -1$

period $= \pi$

$\Rightarrow \boxed{x = \frac{3\pi}{4} + \pi \cdot k,}$ k integer one period

Example 3: Solve the equation in the interval $[0,\pi)$: $2\sin(2x)=1$

$2\sin(2x) = 1$

$\sin(2x) = \frac{1}{2} \Rightarrow \frac{2x}{2} = \frac{\pi/6}{2}$ or $\frac{2x}{2} = \frac{5\pi/6}{2}$

period $= \frac{2\pi}{2} = \pi$

$\Rightarrow \boxed{x = \frac{\pi}{12} \quad \text{or} \quad x = \frac{5\pi}{12}}$

Example 4: Solve the equation in the interval $[0,2\pi)$: $\csc^2 x = 4$ ← In one period

$\csc^2 x = 4 \iff \csc x = +2$ or $\csc x = -2$

$\csc x = \pm 2$

$\frac{1}{\sin x} = 2$ | $\frac{1}{\sin x} = -2$

$\sin x = \frac{1}{2}$ | $\sin x = \frac{-1}{2}$

$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{or} \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

Example 5: Find all solutions to the equation: $\cos(2x)=0$

need to find solutions in one period first and add repetitions of periods.

$\to \cos(2x) = 0 \Rightarrow \frac{2x}{2} = \frac{\pi/2}{2}$ or $\frac{2x}{2} = \frac{3\pi/2}{2}$

period $= \frac{2\pi}{2} = \pi$

$x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$

All solutions: $\boxed{\begin{aligned} &x = \frac{\pi}{4} + \pi\cdot k, \quad k \text{ integer.} \\ &x = \frac{3\pi}{4} + \pi\cdot k \end{aligned}}$

one period

Example 6: Solve the equation in the interval $[0,2\pi)$: $2\sin^2 x - 5\sin x - 3 = 0$

$$2\sin^2 x - 5\sin x - 3 = 0$$

$$\underbrace{(\sin x - 3)}_{0}\underbrace{(2\sin x + 1)}_{0} = 0$$

$\cancel{\sin x - 3 = 0}$ or $2\sin x + 1 = 0$

$\cancel{\sin x = 3}$ $\quad 2\sin x = -1$

Can't happen $\quad \sin x = -\frac{1}{2}$

$\Longrightarrow$ one period $\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

Example 7: Solve the equation in the interval $[0,2\pi)$: $\cos^2 x - 3\sin x - 3 = 0$

$\underbrace{\cos^2 x} - 3\sin x - 3 = 0$ (Transform in an equation with just one trig. function)

$1 - \sin^2 x - 3\sin x - 3 = 0$

$-\sin^2 x - 3\sin x - 2 = 0$

$\sin^2 x + 3\sin x + 2 = 0$

$\underbrace{(\sin x + 1)}_{0}$ or $\underbrace{(\sin x + 2)}_{0} = 0$

$\sin x + 1 = 0$ or $\cancel{\sin x + 2 = 0}$

$\sin x = -1$ $\quad \cancel{\sin x = -2}$

$\boxed{x = \frac{3\pi}{2}}$ $\quad$ Can't happen

Example 8: Solve the equation in the interval $[0,2\pi)$: $\cos(2x) = 5\sin^2 x - \cos^2 x$

$\underbrace{\cos(2x)}_{2\cos^2 x - 1} = 5\underbrace{\sin^2 x}_{1-\cos^2 x} - \cos^2 x$ (Always, keep just one trig. expression.)

(We'll do everything with $\underline{\cos x}$)

$\Longrightarrow 2\cos^2 x - 1 = 5(1 - \cos^2 x) - \cos^2 x$

$2\cos^2 x - 1 = 5 - 5\cos^2 x - \cos^2 x = 5 - 6\cos^2 x$

$+6\cos^2 x \quad +1 \qquad\qquad +1 \quad +6\cos^2 x$

$8\cos^2 x = 6 \iff \cos^2 x = \frac{6}{8} \iff \cos^2 x = \frac{3}{4}$

$\cos x = \pm\sqrt{\frac{3}{4}} \longrightarrow \cos x = +\frac{\sqrt{3}}{2}$ or $\cos x = -\frac{\sqrt{3}}{2}$

$\boxed{x = \frac{\pi}{6}, \frac{11\pi}{6} \text{ or } x = \frac{5\pi}{6}, \frac{7\pi}{6}}$

one period

Example 8: Solve the equation in the interval $[0,2\pi)$: $\cos(2x) = 5\sin^2 x - \cos^2 x$

- Transform into an expression with only one trigonometric function if possible.
- We'll do everything with $\sin(x)$

$$\underbrace{\cos(2x)}_{1-2\sin^2 x} = 5\sin^2 x - \underbrace{\cos^2 x}_{(1-\sin^2 x)}$$

$$1-2\sin^2 x = 5\sin^2 x - (1-\sin^2 x)$$

$$1 - \cancel{2\sin^2 x} = 5\sin^2 x \cancel{-1} + \sin^2 x$$

$$+1 \quad \cancel{+2\sin^2 x} \qquad \cancel{+1} \quad +2\sin^2 x$$

$$2 = 8\sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{2}{8} = \frac{1}{4} \Rightarrow \sin x = \pm\frac{1}{2}$$

$\sin x = \frac{1}{2}$ or $\sin x = -\frac{1}{2}$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{or} \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}}$$

Popper #29

① A ② B ③ C ④ D

To be continued on Friday, 04/15

Example 9: Find all solutions to the equation: $\sin^2 x\cos x = \cos x$

(there is a common factor, bring in one side)

$\sin^2 x \cdot \cos x = \cos x$

$\sin^2 x \cos x - \cos x = 0$

$\underbrace{\cos x}_{0}\underbrace{(\sin^2 x - 1)}_{0} = 0$

$\cos x = 0$ or $\sin^2 x = 1 \Rightarrow \sin x = \pm 1$

$\underbrace{x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \frac{\pi}{2}, \frac{3\pi}{2}}_{\text{in one period}}$

All Solutions:

$$\boxed{\begin{aligned} x &= \frac{\pi}{2} + 2\pi \cdot k, \\ x &= \frac{3\pi}{2} + 2\pi \cdot k, \end{aligned}}$$

k integer

Example 10: Find all solutions: $\sec^2 x + 2\tan x = 0$

(use $\sec^2 x = 1 + \tan^2 x$)

$\sec^2 x + 2\tan x = 0$

$\tan^2 x + 1 + 2\tan x = 0 \iff (\tan x + 1)(\tan x + 1) = 0$

i.e. $\tan x + 1 = 0$

$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}$ in one period $= \pi$

$\Rightarrow \boxed{x = \frac{3\pi}{4} + \pi \cdot k,}$ k integer.

Another version $\Rightarrow$ $\tan x = -1 \Rightarrow x = -\frac{\pi}{4}$ ← Solution

Note that $-\frac{\pi}{4} = \frac{3\pi}{4} + \pi \cdot (-1)$

$\Rightarrow \boxed{x = -\frac{\pi}{4} + \pi \cdot k,}$ k integer

Both equivalent

there are two periodical intervals

Example 11: Solve the equation in the **interval** $[0,2)$: $\cot(\pi x) = -1$

$\cot(\pi x) = -1$

period $= \frac{\pi}{\pi} = 1$

$\frac{\cancel{\pi} x}{\cancel{\pi}} = \frac{3\cancel{\pi}}{4\cancel{\pi}}$ ⟵ over one period

$x = \frac{3}{4} \Rightarrow x = \frac{3}{4} + 1 = \frac{7}{4}$

next period.

$\Rightarrow \boxed{x = \frac{3}{4}, \frac{7}{4}}$

Example 12: Find all solutions of the equation in the interval $[0, 4\pi)$: $2\sin\left(\frac{x}{2}\right) = 1$

one period

$2\sin\left(\frac{x}{2}\right) = 1$

$\Rightarrow \sin\left(\frac{x}{2}\right) = \frac{1}{2} \Rightarrow$ Think, in one full rotation,

period $= \frac{2\pi}{1/2} = 4\pi$

$2 \times \frac{x}{2} = \frac{\pi}{6} \times 2$ or $2 \times \frac{x}{2} = \frac{5\pi}{6} \times 2$

$x = \frac{\cancel{2}\pi}{\cancel{6}} = \frac{\pi}{3}$, $x = \frac{\cancel{10}^{5}\pi}{\cancel{6}_{3}} = \frac{5\pi}{3}$

$\Rightarrow \boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$

extra example:

Solve $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ in $\underbrace{[0, 8\pi)}_{\text{2 periods}}$:

Solution: $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$

$\left(\text{period} = \frac{2\pi}{1/2} = 4\pi\right)$

$\cancel{2}\cdot\frac{x}{\cancel{2}} = \frac{\pi}{6}\cdot 2$ or $\cancel{2}\cdot\frac{x}{\cancel{2}} = \frac{5\pi}{6}\cdot 2$

$x = \frac{2\pi}{6} = \frac{\pi}{3}$ or $x = \frac{10\pi}{6} = \frac{5\pi}{3}$

i.e. $x = \frac{\pi}{3}, \frac{5\pi}{3}$ in one period

↳ extend to next period:

$x = \underbrace{\frac{\pi}{3} + \frac{4\pi\cdot 3}{1\cdot 3}}_{\frac{13\pi}{3}}$, $x = \underbrace{\frac{5\pi}{3} + \frac{4\pi\cdot 3}{1\cdot 3}}_{\frac{17\pi}{3}}$

$\Rightarrow$ $\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \text{ in } [0, 8\pi)}$

Note: Pay Attention to the Interval.

Always, use identities (if possible) to simplify!

Example 13: Find all solutions of the equation in the interval $[0, 2\pi)$: $\sec(x + 2\pi) = 2$

period $= 2\pi$

Hence, $\sec(\underbrace{x+2\pi}_{\text{period}}) = \sec(x)$

Thus, $\sec(x) = 2$

$$\frac{1}{\cos x} = 2$$

$$\cos x = \frac{1}{2} \Longrightarrow \boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$$

Example 14: Find all solutions of the equation in the interval $[0, \pi)$: $2\sin\left(2x - \frac{3\pi}{2}\right) = \sqrt{2}$

one period

$$\Rightarrow 2\sin\left(2x - \frac{3\pi}{2}\right) = \sqrt{2}$$

$$\sin\left(2x - \frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}$$

$\frac{\pi}{4}$ or $\frac{3\pi}{4}$

(there is no identity to apply, hence go to unit circle.)

In one full rotation, this expression

$$2x - \frac{3\pi}{2} = \frac{\pi}{4} \quad \text{or} \quad 2x - \frac{3\pi}{2} = \frac{3\pi}{4}$$

$$2x = \frac{\pi}{4} + \frac{3\pi}{2}\cdot\frac{2}{2} \qquad 2x = \frac{3\pi}{4} + \frac{3\pi}{2}\cdot\frac{2}{2}$$

$$\frac{2x}{2} = \frac{\frac{7\pi}{4}}{2} \qquad \frac{2x}{2} = \frac{\frac{9\pi}{4}}{2}$$

$$\Rightarrow \boxed{x = \frac{7\pi}{8}} \leftarrow \text{Answer.} \qquad x = \frac{9\pi}{8} \text{ not in } [0, \pi)$$

Make sure you recognize formulas:

$$\sin^2\left(\underbrace{\tfrac{3\pi}{7}}_{x}\right) + \cos^2\left(\underbrace{\tfrac{3\pi}{7}}_{x}\right) = 1$$ ← No Hesitation

match!

Popper # 30

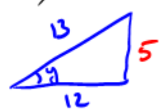
(1) $\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \cos\left(2 \times \frac{\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

A. 0 B. 1 C. $\frac{1}{2}$ D. $\frac{\sqrt{2}}{2}$ E. none

Given $\sin(x) = \frac{3}{5}$, $\cos(y) = \frac{12}{13}$, $0 < x < 90^\circ$, $270^\circ < y < 360^\circ$.

$\cos x = \frac{4}{5}$

$\sin(y) = -\frac{5}{13}$



(2) $\sin(x+y) = \sin x \cos y + \cos x \sin y = \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \left(-\frac{5}{13}\right) = \frac{16}{65}$

A. $\frac{36}{65}$ B. $\frac{16}{65}$ C. $-\frac{16}{65}$ D. $\frac{-20}{65}$

(3) $\tan\left(\frac{y}{2}\right) = \frac{1-\cos y}{\sin y} = \frac{1 - \frac{12}{13}}{\frac{-5}{13}} = \frac{\frac{1}{\cancel{13}}}{\frac{-5}{\cancel{13}}} = -\frac{1}{5}$

A. $-\frac{12}{5}$ B. $-\frac{1}{5}$ C. $\frac{5}{13}$ D. none

(4) $\cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

A. $\frac{7}{25}$ B. $\frac{18}{25}$ C. $-\frac{7}{25}$ D. none