Right Triangle: If you know one side, one angle, or two sides, you can find the missing pieces.

Well, we'll do the same for any triangle as long as we know 3 important piece of info:

- two angles, one side,
- three sides
- two sides, one angle

Section 7.3 - The Law of Sines and the Law of Cosines

Sometimes you will need to solve a triangle that is not a right triangle. This type of triangle is called an oblique triangle. To solve an oblique triangle you will not be able to use right triangle trigonometry. Instead, you will use the Law of Sines and/or the Law of Cosines.

You will typically be given three parts of the triangle and you will be asked to find the other three. The approach you will take to the problem will depend on the information that is given.

If you are given SSS (the lengths of all three sides) or SAS (the lengths of two sides and the measure of the included angle), you will use the Law of Cosines to solve the triangle.

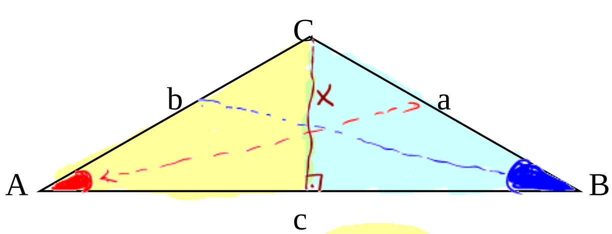
If you are given SAA (the measures of two angles and one side) or SSA (the measures of two sides and the measure of an angle that is not the included angle), you will use the Law of Sines to solve the triangle.

Recall from your geometry course that SSA does not necessarily determine a triangle. We will need to take special care when this is the given information.

⟹ We need two technical results:

1 THE LAW OF SINES

Here's the **Law of Sines**. In any triangle *ABC*,



yellow △ : $\sin A = \frac{x}{b}$

$\Rightarrow x = b \sin A$

Blue △ : $\sin B = \frac{x}{a}$

$\Rightarrow x = a \sin B$

Combine

$$\frac{\cancel{b}\sin A}{a\cancel{b}} = \frac{\cancel{a}\sin B}{\cancel{a}b}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Correspondence Law

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}.$$

USED FOR SAA, SSA cases!

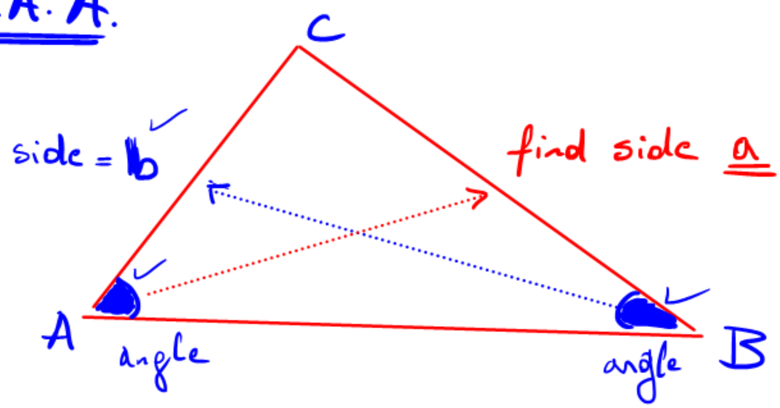
SAA: One side and two angles are given

SSA: Two sides and an angle opposite to one of those sides are given

Law of Sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

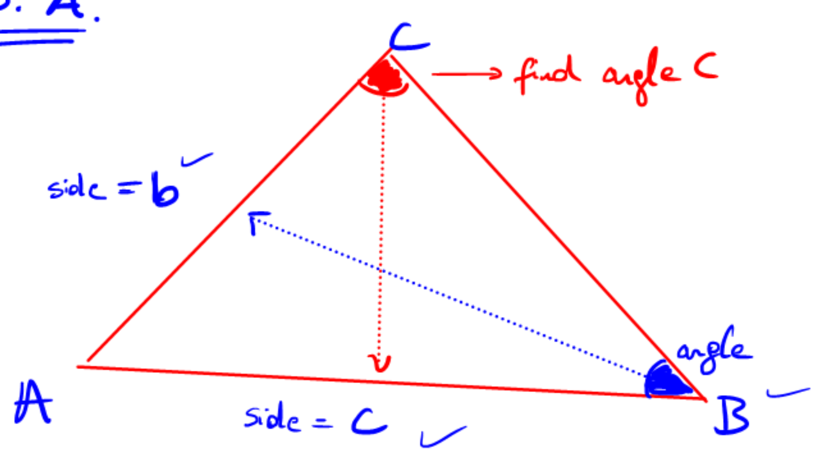
S.A.A.



$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

find this!

S.S.A.



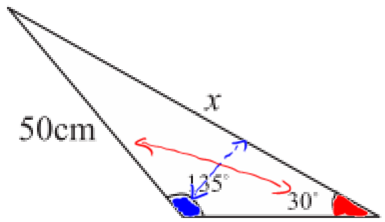
find $\hat{C}$

$$\frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Law of Sines: SAA → find side, SSA → find angle.

Example 1: Find x.

SAA



Law of Sines

$$\frac{\sin(135^\circ)}{x} = \frac{\sin(30^\circ)}{50} \leftarrow \text{cross-product}$$

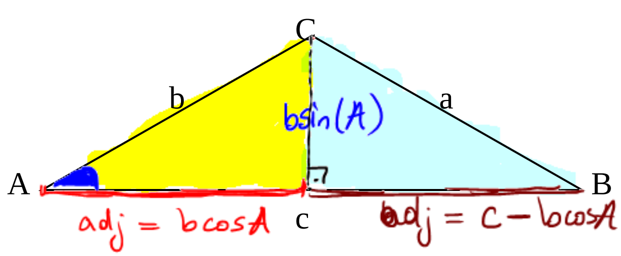
$$x \cdot \sin(30^\circ) = 50 \cdot \sin(135^\circ) \leftarrow \text{divide by } \sin(30^\circ)$$

$$\Rightarrow \quad x = \frac{50 \cdot \sin(135^\circ)}{\sin(30^\circ)} \underset{\text{evaluate}}{=} \frac{50 \cdot \frac{\sqrt{2}}{2}}{\frac{1}{2}} = 50\sqrt{2}$$

$$\boxed{x = 50\sqrt{2} \text{ cm}}$$

⇒ **THE LAW OF COSINES**

Here's the **Law of Cosines.** In any triangle *ABC*,



In yellow △, opp = b sinA
adj = b cosA

In blue △, use Pythagorean

$$(b\sin A)^2 + (c - b\cos A)^2 = a^2$$

⋮ simplify

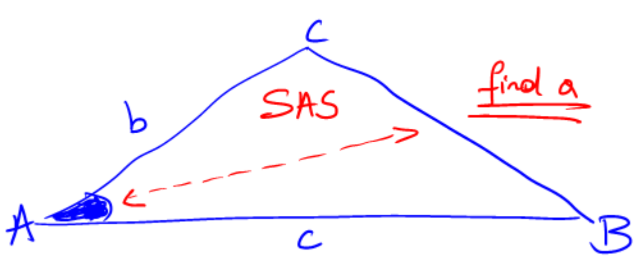
$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

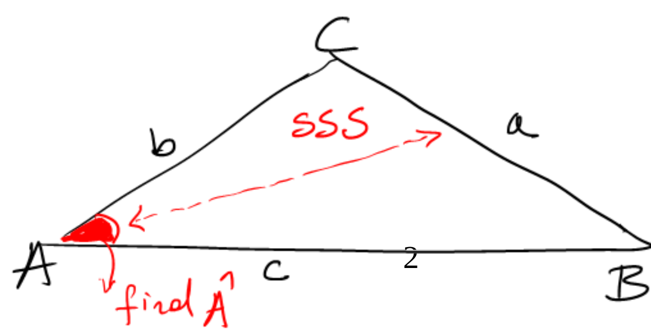
Pythagorean type

$$\left\{\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}\right.$$

USED FOR SAS , SSS cases!

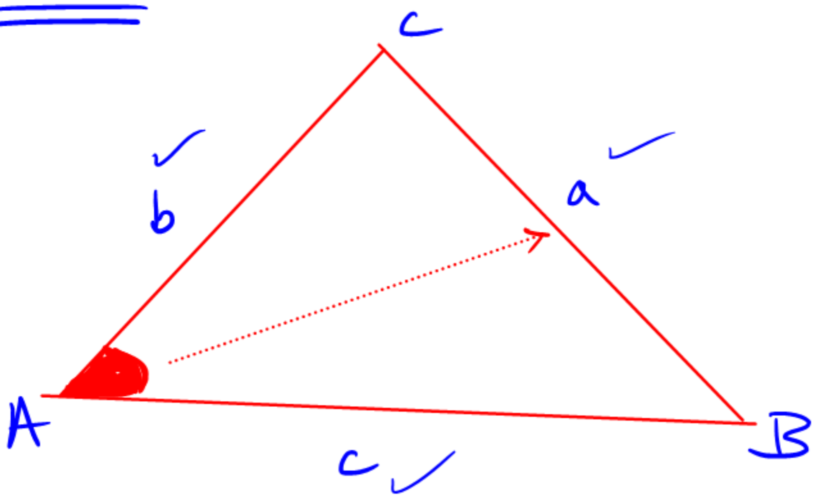
SAS: Two sides and the included angle are given
SSS: Three sides are given





Law of Cosines : "Pythagorean Type"

S.S.S

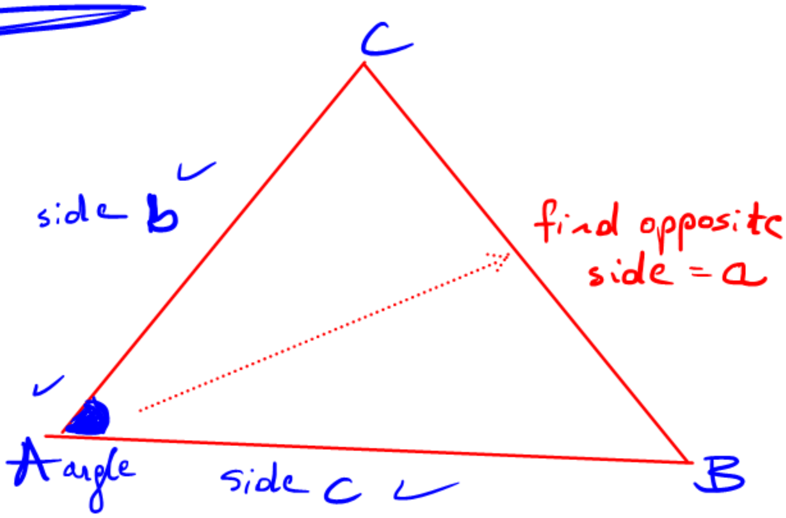


→ find angles

$$a^2 = b^2 + c^2 - 2\cdot b\cdot c\cdot \cos A$$

solve for $\underline{\cos(A)}$

S.A.S.



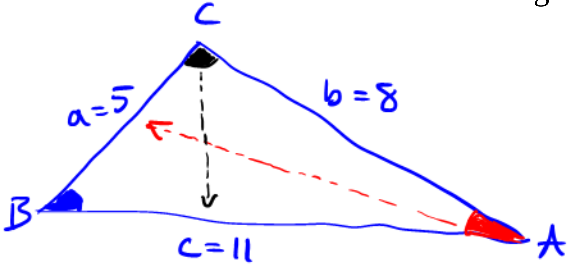
$$a^2 = b^2 + c^2 - 2bc\cos A$$

↳ take square root!

Law of Cosines: SSS → find angles, SAS → find side

SSS

Example 2: In ΔABC, $a = 5$, $b = 8$, and $c = 11$. Find the measures of the three angles to the nearest tenth of a degree.



$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$5^2 = 8^2 + 11^2 - 2\cdot 8\cdot 11\cdot \cos A$$

$$25 = 185 - 176\cos A$$

$$-160 = -176 \Rightarrow \cos A = \frac{-160}{-176} = \frac{10}{11}$$

$$\Rightarrow \cos A = \frac{10}{11} \Rightarrow \boxed{\hat{A} = \cos^{-1}\left(\frac{10}{11}\right) = 24.6^\circ}$$

To find $\hat{C}$:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$11^2 = 8^2 + 5^2 - 2\cdot 8\cdot 5\cos C$$

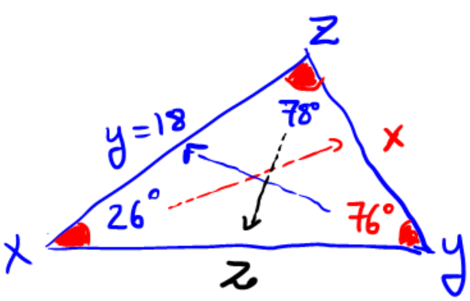
$$121 = 89 - 80\cos C$$

$$32 = -80\cos C$$

$$\cos C = \frac{-32}{80} = -\frac{2}{5} \Rightarrow \boxed{\hat{C} = \cos^{-1}\left(-\frac{2}{5}\right) = 113.6^\circ}$$

$$\boxed{\hat{B} = 180^\circ - (\hat{A} + \hat{C}) = 41.8^\circ}$$

Example 3: In ΔXYZ, $\angle X = 26°$, $\angle Z = 78°$ and $y = 18$. Solve the triangle. Give exact answers.



- Find angle $\boxed{\hat{y} = 180^\circ - (26^\circ + 78^\circ) = 76^\circ}$

- To find $\underline{x}$: SSA → find side

$$\frac{\sin 76^\circ}{18} = \frac{\sin 26^\circ}{x}$$

$$x \cdot \frac{\cancel{\sin 76^\circ}}{\cancel{\sin 76^\circ}} = 18\,\frac{\sin 26^\circ}{\sin 76^\circ} \Rightarrow \boxed{x = \frac{18\sin 26^\circ}{\sin 76^\circ}}$$

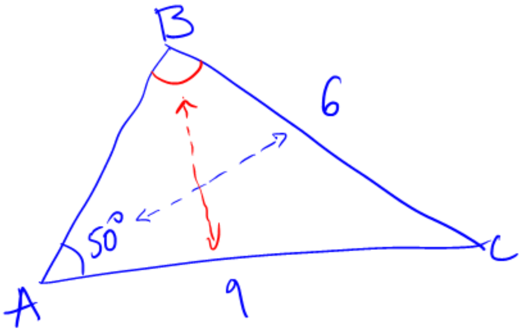
- To find $\underline{z}$: SSA → find side

$$\frac{\sin 78^\circ}{z} = \frac{\sin 76^\circ}{18} \Rightarrow \boxed{z = \frac{18\cdot\sin 78^\circ}{\sin 76^\circ}}$$

$$z\cdot\frac{\sin 76^\circ}{\sin 76^\circ} = 18\cdot\frac{\sin 78^\circ}{\sin 76^\circ}$$

To be continued on Monday, 04/25

Example 4: In ΔABC, $\angle A = 50°$, $b = 9$ and $a = 6$. Solve the triangle and round all answers to the nearest hundredth.



SSA ⟷ Law of Sines
(not in between)

Let's find $\hat{B}$:

$$\frac{\sin(B)}{9} = \frac{\sin(50°)}{6} \Rightarrow \sin(B) = \frac{9\sin(50°)}{6}$$

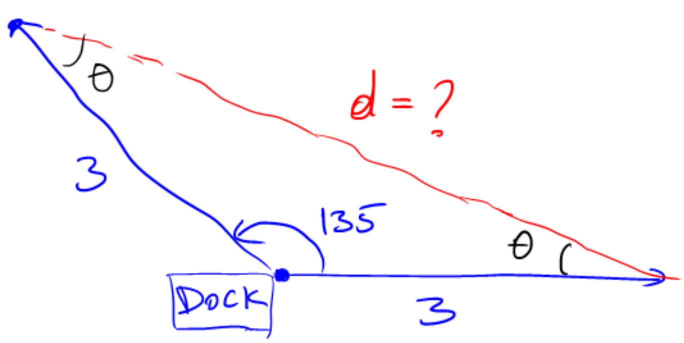
$\sin(B) = 1.15$!!!

$\boxed{\text{IMPOSSIBLE}}$

There is no such a triangle.

Example 5: Two sailboats leave the same dock together traveling on courses that have an angle of 135° between them. If each sailboat has traveled 3 miles, how far apart are the sailboats from each other?

SAS



LAW of Cosines :

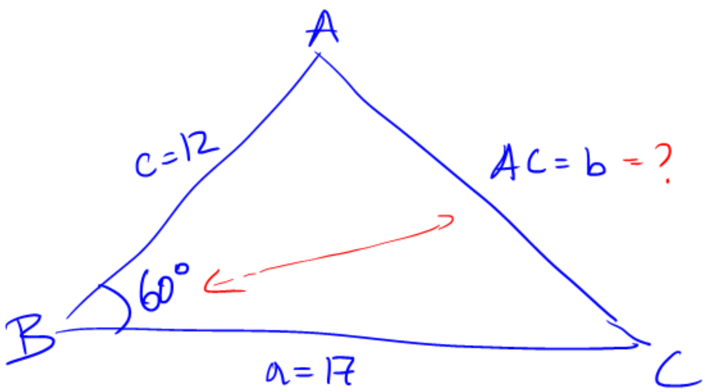
$$d^2 = 3^2 + 3^2 - 2\cdot 3\cdot 3\cdot\cos(135°)$$

$$d^2 = 18 + 18\frac{\sqrt{2}}{2} = 18 + 9\sqrt{2}$$

$$\boxed{d = \sqrt{18 + 9\sqrt{2}} = \sqrt{9(2+\sqrt{2})} = 3\sqrt{2+\sqrt{2}}}$$

← exam type question

Example 6: In ΔABC, $\angle B = 60°$, $a = 17$ and $c = 12$. Find the length of AC.



$$b^2 = a^2 + c^2 - 2ac\cdot\cos 60^\circ$$

$$b^2 = 17^2 + 12^2 - \cancel{2}\cdot 17\cdot 12\cdot \frac{1}{\cancel{2}}$$

$$b^2 = 289 + 144 - 204$$

$$\boxed{b = \sqrt{229}}$$

$\left.\begin{array}{l}\text{S.A.S}\\ \text{S.S.S}\\ \text{A.S.A}\end{array}\right\} \Rightarrow$ One "possible" triangle

Note: SSA case is called the ambiguous case of the law of sines. There may be two solutions, one solution, or no solutions. You should throw out the results that don't make sense. That is, if $\sin A > 1$ or the angles add up to more than 180^0.

SSA Case (Two sides and an angle opposite to those sides)

In the last case (SSA) for solving oblique triangles, two sides and the angle opposite one of those sides are given. Suppose that θ is the given angle. The other given side must be adjacent to θ. We consider several cases.

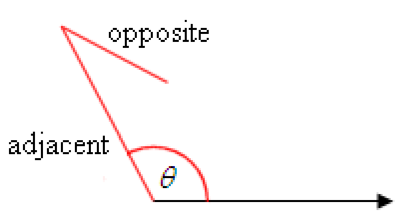
Case 1: Suppose that $\theta > 90^\circ$. Two possibilites arise.

(a) If opposite $\leq$ adjacent, no triangle is formed. (There is no solution.)

(b) If opposite $>$ adjacent, one triangle is formed. (Use the Law of Sines.)

Case 1 Part (a)

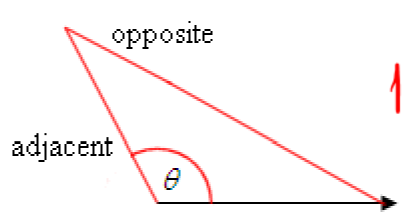
No triangle



$\theta > 90^\circ$

opposite $\leq$ adjacent

Case 1 Part (b)



1 triangle

$\theta > 90^\circ$

opposite $>$ adjacent

SSA - Graph — FOLLOW THE GIVEN INFO STRICTLY

It is the situation when an angle θ is given, and adjacent side, and opposite side.

→ Think of θ:

$\boxed{\theta > 90^\circ}$ wide, then

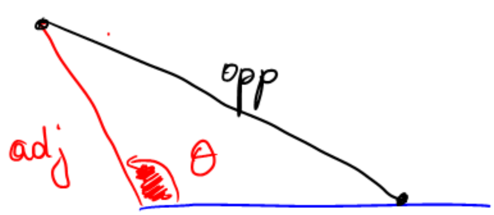
(1)

Compare given sides:

adj > opposite

No triangle formed

(2)



2) opposite > adjacent

1 triangle formed.

If $\theta > 90^\circ$, just compare the sides and you'll know the answer "1 triangle or none"

This is the simple Case.

Case 2: Suppose that $\theta < 90^{\circ}$. Four possibilites arise. Let h be the length of the altitude of the triangle drawn from the vertex that connects the opposite and adjacent sides.

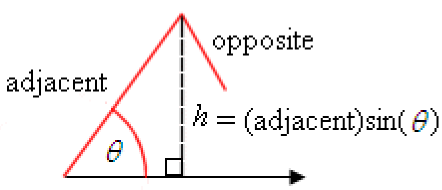
(a) If opposite $< h <$ adjacent, no triangle is formed. (There is no solution.)

(b) If opposite $= h <$ adjacent, one right triangle is formed. (Use right triangle trigonometry to solve the triangle as in Section 7.1.)

(c) If $< h <$ opposite $<$ adjacent, two different triangles are formed. This is called the ambiguous case. (Use the Law of Sines to find two solutions.)

(d) If opposite $\geq$ adjacent, one triangle is formed. (Use the Law of Sines.)

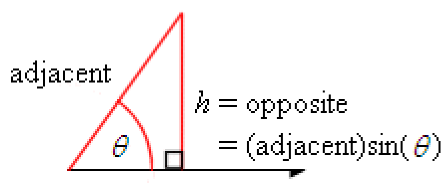
Case 2 Part (a)



$\theta < 90^{\circ}$

opposite $< h <$ adjacent

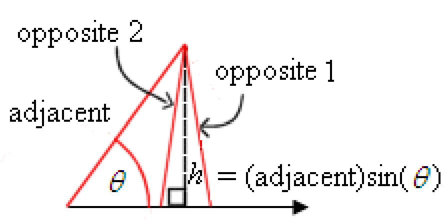
Case 2 Part (b)



$\theta < 90^{\circ}$

opposite $= h <$ adjacent

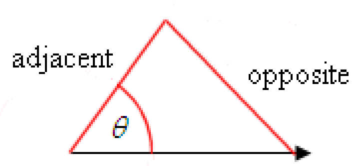
Case 2 Part (c)



$\theta < 90^{\circ}$

$h <$ opposite $<$ adjacent

Case 2 Part (d)



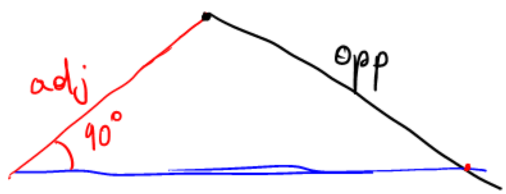
$\theta < 90^{\circ}$

opposite $\geq$ adjacent

SSA, angle – adjacent side – opposite side, Graph

$\boxed{\theta < 90^\circ}$

(1)

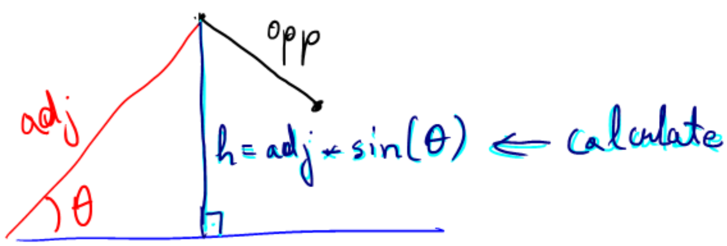


If $\theta > 90^\circ$, and

adj < opposite,

$\boxed{\text{1 triangle formed}}$

(2) adj > opp

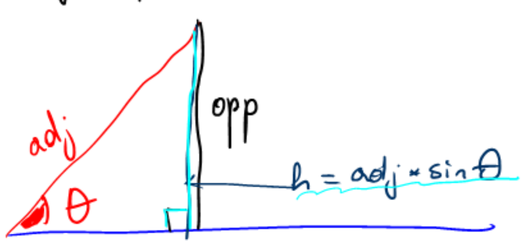


If

adj > $\underline{\underline{h}}$ > opp.

$\boxed{\text{No triangle}}$

(3) adj > opp



If

adj > opp = $\underline{\underline{h}}$

$\boxed{\text{1 triangle formed}}$

(4)

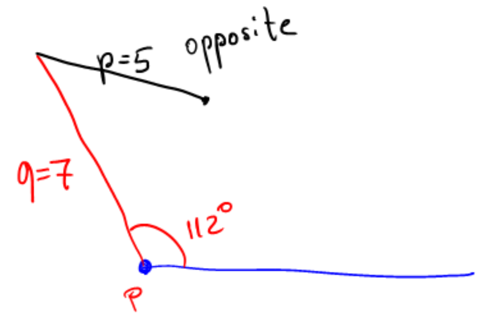
If

adj > opp > h = adj * sinθ

then $\boxed{\text{2 triangle formed}}$

opp adj

Example 7: In ΔPQR, $\angle P = 112°$, $p = 5$ and $q = 7$. How many possible triangles are there? Solve the triangle. Round the answers to three decimal places.



$p=5 < q=7$

opposite < adjacent

$\boxed{\text{No triangle}}$

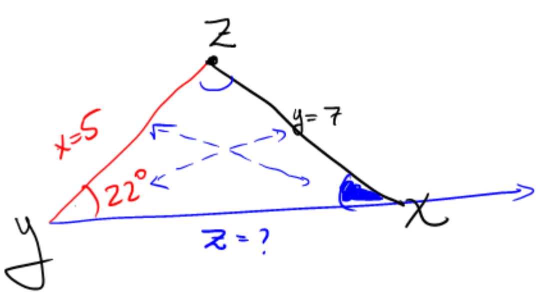
exercise

acute opp adj adjacent > opposite

Example 8: In ΔXYZ, $\angle Y = 22°$, $y = 7$, $x = 5$. How many possible triangles are there? Solve the triangle and round all answers to the nearest hundredth.

1 triangle

$x=5$, $y=7 \Rightarrow \hat{X} < \hat{Y}$



Law of Sines

$$\frac{\sin(X)}{5} = \frac{\sin(22°)}{7}$$

$$\Rightarrow \sin(X) = \frac{5\sin(22°)}{7} \approx 0.27$$

$$\sin X = 0.27 \Rightarrow \begin{cases} \hat{X} = 15.52° \\ \text{or } \hat{X} = 180° - 15.52° \end{cases}$$

Don't forget $\hat{X} < \hat{Y}$

$$\Rightarrow \boxed{\hat{X} \simeq 15.52°} < \boxed{\hat{Y} = 22°}$$

$$\Rightarrow \hat{Z} = 180° - (15.52° + 22°)$$

$$\boxed{\hat{Z} = 142.48°}$$

Law of Sines again

$$\Rightarrow \frac{\sin(142.48°)}{z} = \frac{\sin(22)}{7} \Rightarrow z = \frac{7\sin(142.48°)}{\sin(22°)}$$

$$\boxed{z \approx 11.38}$$