Popper 05

Bubble

Given $f(x) = \dfrac{3x^2 - 3}{x^2 + 2x - 3}$, find

(1) horizontal asymptote $y = \frac{3}{1} = 3$

A. 3 B. 0 C. 1 D. none of them

(2) vertical asymptote $= \dfrac{3\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}}$ $x+3=0$ $x=-3$

A. $x=-1$, $x=1$ B. $x=1$, $x=-3$ C. $x=-3$ D. none of them

$\dfrac{3(x+1)}{x+3}$

(3) hole(s) of f, and their location $x=1$, $f^*(1) = \dfrac{3(1+1)}{1+3}$

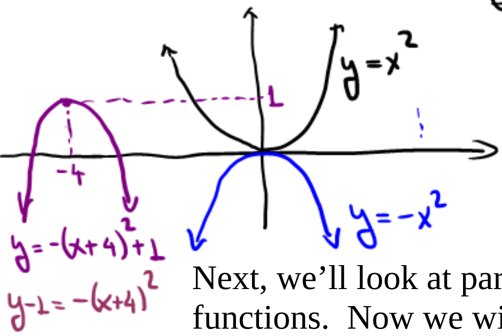
A. $x=-3$, $y = f^*(-3)$ B. $x=1$, $y = f^*(1) = \frac{6}{4}$ C. no holes D. none of them

(4) Bubble A.

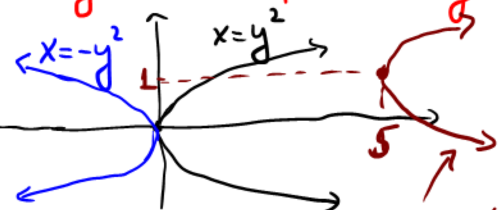
(5) Bubble B.

Parabolas ⟷ we visualize $y=x^2$

We can have "horizontal" parabolas but they are not function anymore.



All functions = "vertical" parabolas.



Next, we'll look at parabolas. We previously studied parabolas as the graphs of quadratic functions. Now we will look at them as conic sections. There are a few differences. For example, when we studied quadratic functions, we saw that the graphs of the functions could open up or down. As we look at conic sections, we'll see that the graphs of these second degree equations can also open left or right. So, not every parabola we'll look at in this section will be a function.

Overall, parabolas rise as a result of $y=x^2$ or $x=y^2$. The rest is transformation

We already know that the graph of a quadratic function $f(x)=ax^2+bx+c$ is a parabola. But there is more to be learned about parabolas.

$f(x)=ax^2+bx+c$

$\Rightarrow y=a(x-h)^2+k \implies \frac{y-k}{a}=\frac{a}{a}(x-h)^2 \implies \boxed{\frac{y-k}{a}=(x-h)^2}$

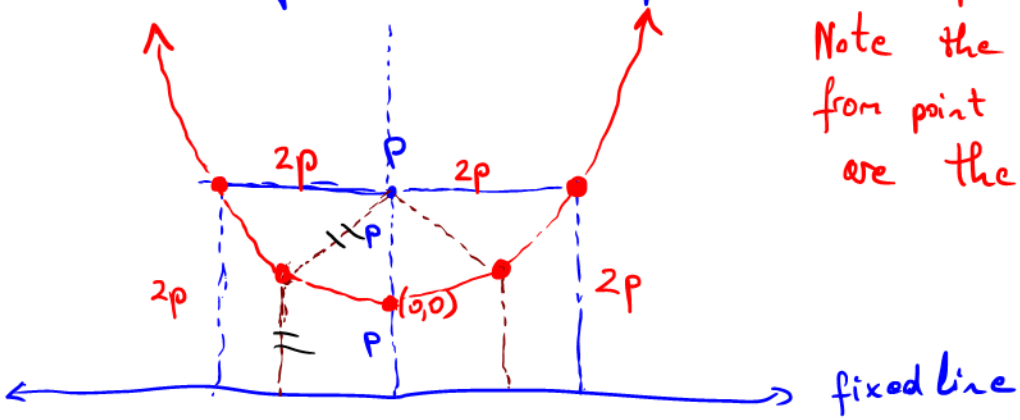
$(h,k)=$ Vertex

Always, we'll leave the square alone.

Definition: A *parabola* is the set of all points equally distant from a fixed line and a fixed point not on the line. The fixed line is called the *directrix*. The fixed point is called the *focus*.

Understand definition: Put some points = red dot point

Note the distances from point P and line are the same.

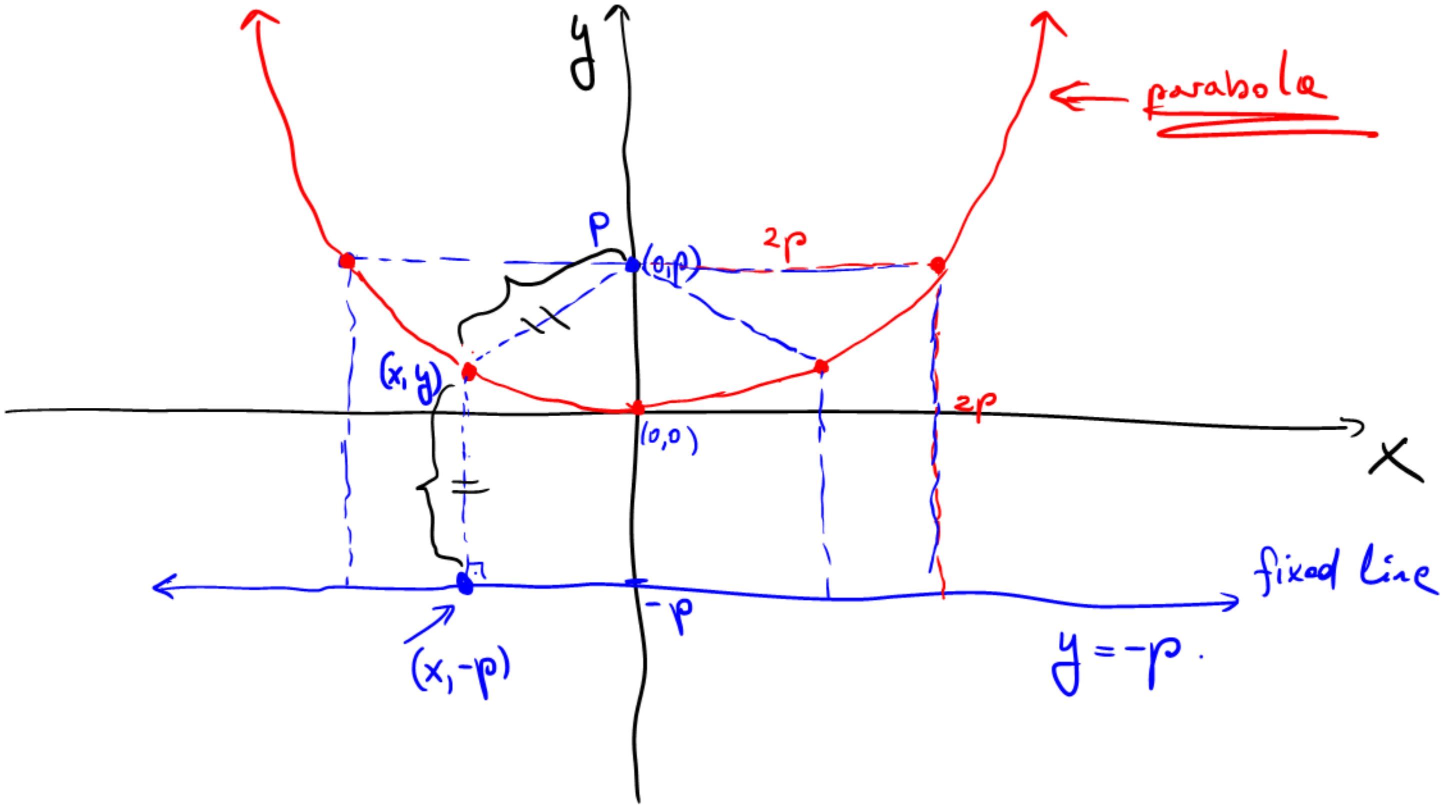


Next step: How to get a formula using this information?

The *axis*, or *axis of symmetry*, runs through the focus and is perpendicular to the directrix.

The *vertex* is the point **halfway between** the focus and the directrix.

We won't be working with slanted parabolas, just with "horizontal" and "vertical" parabolas.



In this setting, vertex is $(0,0)$.
halfway of distance from P to line.

Let (x,y) be any point in Parabola, by definition

distance $(x,y) \Leftrightarrow (0,p)$ = distance $(x,y) \Leftrightarrow$ line's point $(x,-p)$

$$(x-0)^2 + (y-p)^2 = (x-x)^2 + (y+p)^2$$

$$x^2 + \cancel{y^2} - 2yp + \cancel{p^2} = 0^2 \cancel{+y^2} + 2yp + \cancel{p^2}$$

$$\Rightarrow \boxed{x^2 = 4py}$$

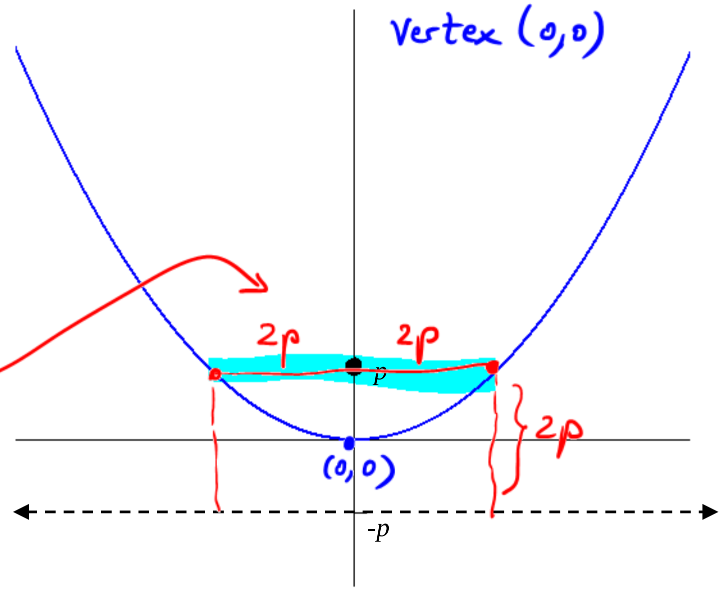
We left square term alone.

vertical

Note: As an exercise, in a similar way, find that formula for horizontal parabolas is $\boxed{y^2 = 4px.}$

$\Rightarrow$ From now on: get used to this new definition.

Basic "Vertical" Parabola: = functions



Equation: $\boxed{x^2 = 4py}$

we can find

Focus: $(0, p)$

Directrix: $y = -p$

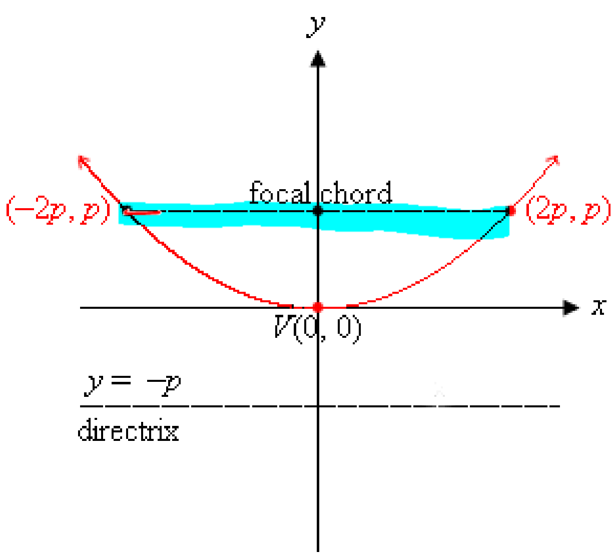
Focal Width: $|4p|$

It is important because it gives the opening of parabola

Note: This can be written as $y = f(x) = \dfrac{x^2}{4p}$. *It is a function (passes vertical line test).*

Do not forget $\boxed{x^2 = 4py \iff y = \dfrac{x^2}{4p} \text{ a function.}}$

The line segment that passes through the focus and perpendicular to the axis with endpoints on the parabola is called the focal chord. Its length (called the focal width) is $4p$.



vertical parabolas with vertex (0,0)

Example: Graph the parabola $x^2 - 16y = 0$.

Steps:

① If equation contains x^2, then it is a vertical parabola.

$$x^2 - 16y = 0 \iff \text{vertical} \qquad \left(\text{It is a function } y = \frac{x^2}{16}\right)$$

② Bring it in standard form. $\boxed{x^2 = 4p\,y}$

$$\boxed{x^2 = 16\,y}$$

③ Find the focus point

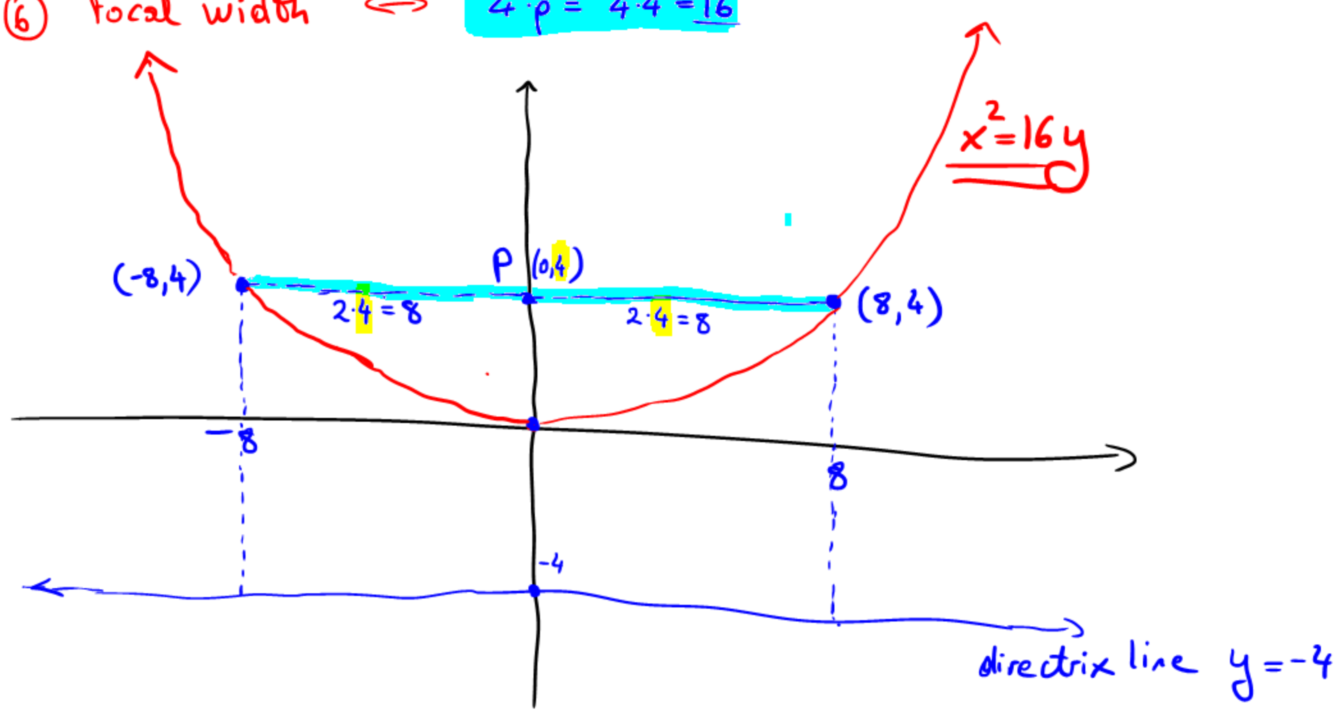
$$16 = 4p \Rightarrow p = 4 \Rightarrow \text{Focus } (0,4) \text{ on } y\text{-axis.}$$

④ Give directrix line:

$$y = -p = -4 \Rightarrow \boxed{y = -4}$$

⑤ Vertex $\leftrightarrow$ $(0,0)$

⑥ Focal width $\leftrightarrow$ $4 \cdot p = 4 \cdot 4 = 16$



Note: You have this steps for parabolas of vertex (0,0) on page 4.

Doing same calculations, but horizontally i.e point P is on x-axis and fixed line is vertical, we get

Basic "Horizontal" Parabola:

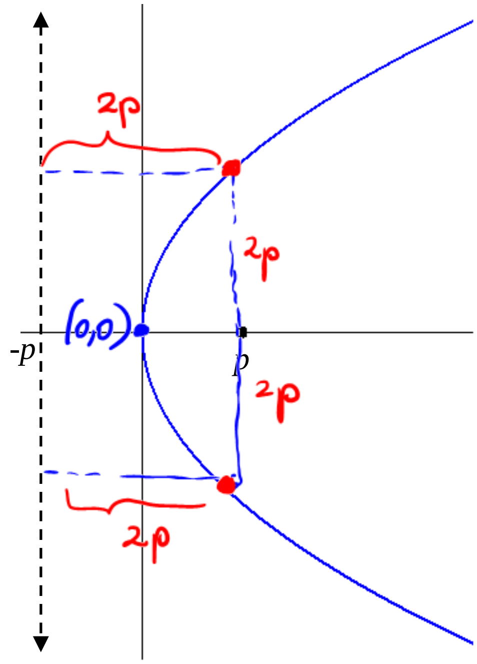
Equation: $\boxed{y^2 = 4px}$

Focus: $(p, 0)$ on x-axis

Directrix: $x = -p$ vertical line

Focal Width: $|4p|$

Vertex $(0,0)$



Note: This is not a function (fails vertical line test). However, the top half $y = \sqrt{x}$ is a function and the bottom half $y = -\sqrt{x}$ is also a function.

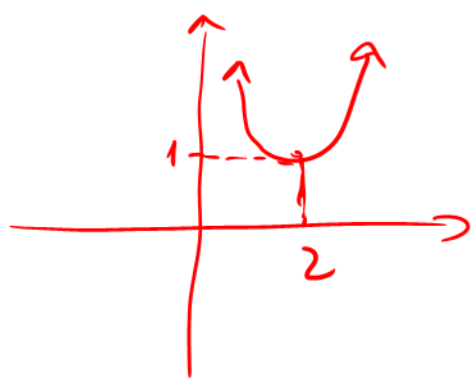
Never forget, a horizontal parabola is never a function.

To continue on Friday, 02/12.

Graphing parabolas with vertex at the origin:

- When you have an equation, look for x^2 or y^2
- If it has x^2, it's a "vertical" parabola. If it has y^2, it's a "horizontal" parabola.
- Rearrange to look like $y^2 = 4px$ or $x^2 = 4py$. In other words, isolate the squared variable.
- Determine *p*.
- Determine the direction it opens.
    - If *p* is positive, it opens right or up.
    - If *p* is negative, it opens left or down.
- Starting at the origin, place the focus *p* units to the inside of the parabola. Place the directrix *p* units to the outside of the parabola.
- Use the focal width $4p$ ($2p$ on each side) to make the parabola the correct width at the focus.

$y^2=4px$ or $x^2=4px$ $\Longleftrightarrow$ parabolas with vertex $(0,0)$



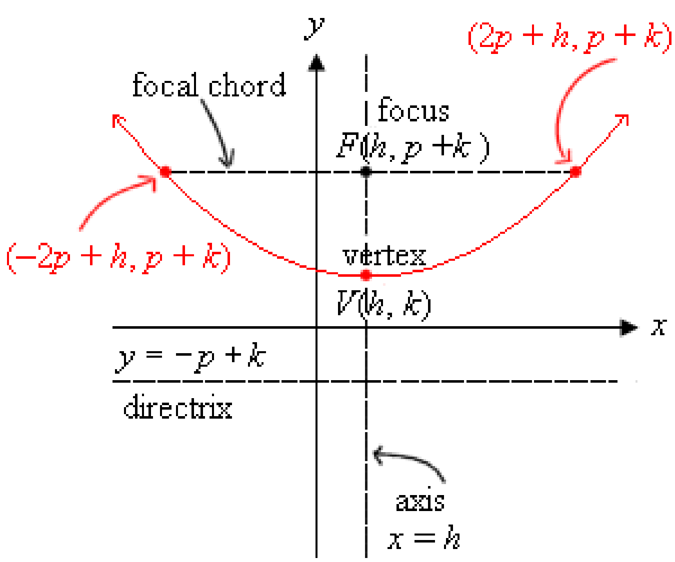
$\hookrightarrow y=(x-2)^2+1$ ← 1 unit up

2 units right

$\hookrightarrow (x-2)^2=(y-1)$

Graphing parabolas with vertex not at the origin:

- Rearrange (complete the square) to look like $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$.
- Vertex is (h,k). Draw it the same way, except start at this vertex.



Graph of the parabola $(x-h)^2 = 4p(y-k)$.

What to keep in mind:

- $(y-k)^2 = 4p(x-h) \leftarrow$ to graph this, is same as $y^2 = 4px$, by shifting the vertex $(0,0)$ to (h,k).
- $(x-h)^2 = 4p(y-k) \leftarrow$ to graph this, is same as $x^2 = 4px$ by shifting the vertex $(0,0)$ to (h,k).

How the transformations work

you shift $y=x^2$
2 units right and
5 units up.

Think of a quadratic function: $y=(x-2)^2+5$

Bring it in a standard form for a parabola definition:

$$(x-2)^2 = (y-5)$$

you do the same here
but in a parabolic point of view.

Thus, you get $x^2=y$ and
move 2 to the right and 5 up.

- Vertex was $(0,0) \longrightarrow (2,5)$.

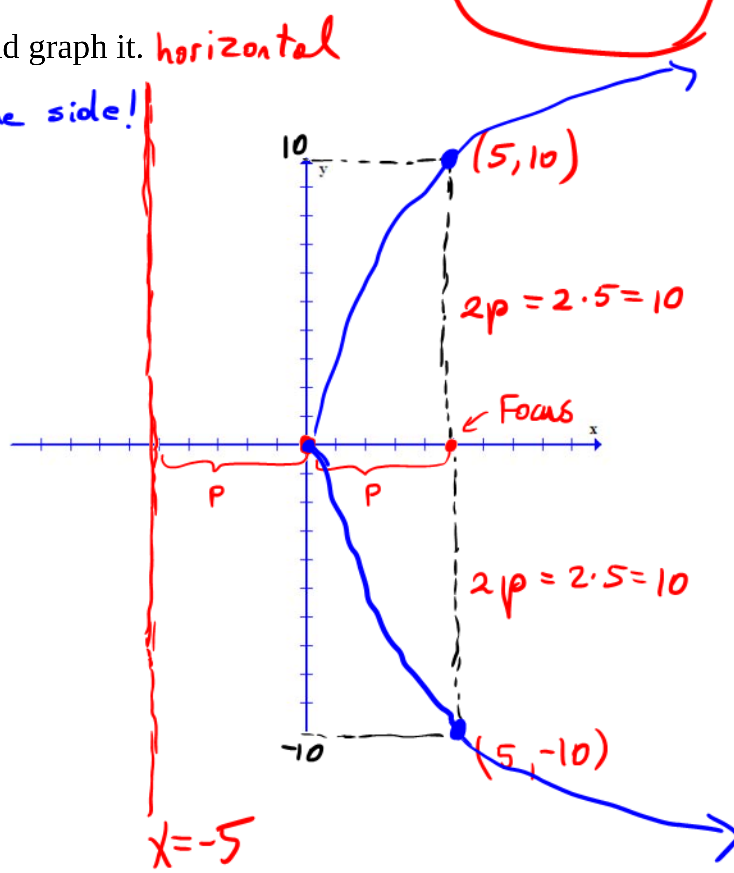
- everything else shifts accordingly:
  2 units right
  5 units up.

• Parabolas with vertex (0,0) $\longleftrightarrow$ $x^2 = 4py$ or $y^2 = 4px$
vertical, horizontal

Example 1: Write $y^2 - 20x = 0$ in standard form and graph it. horizontal

Always leave "square" term in one side!

$$\left.\begin{aligned} y^2 &= 20x \\ &= 4p\,x \end{aligned}\right\} \Rightarrow \begin{aligned} 4p &= 20 \\ p &= 5 \end{aligned}$$



Vertex: (0,0)

Focus: (5,0)

Directrix: opposite side of focus
$x = -5$

Focal width: $= 4p = 4 \cdot 5 = 20$
→ graph a line passing through focus.

Endpoints of focal chord:

From construction:

$(5, 10)$ and $(5, -10)$

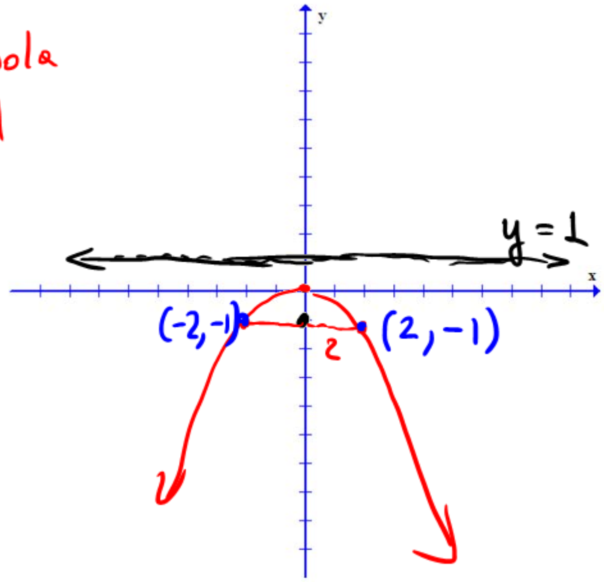
$(\underset{p}{5}, \underset{2p}{10})$ and $(\underset{p}{5}, \underset{-2p}{-10})$

Example 2: Write $6x^2 + 24y = 0$ in standard form and graph it.

$x^2 = 4py$

$$\frac{6x^2}{6} = \frac{-24y}{6}$$

$x^2 = -4y \Rightarrow 4p = -4 \Rightarrow p = -1$ } Vertical parabola down ward



Vertex: $(0,0)$

Focus: $(0,-1)$

Directrix: $y = +1$

Focal width: $4 \cdot 1 = 4$

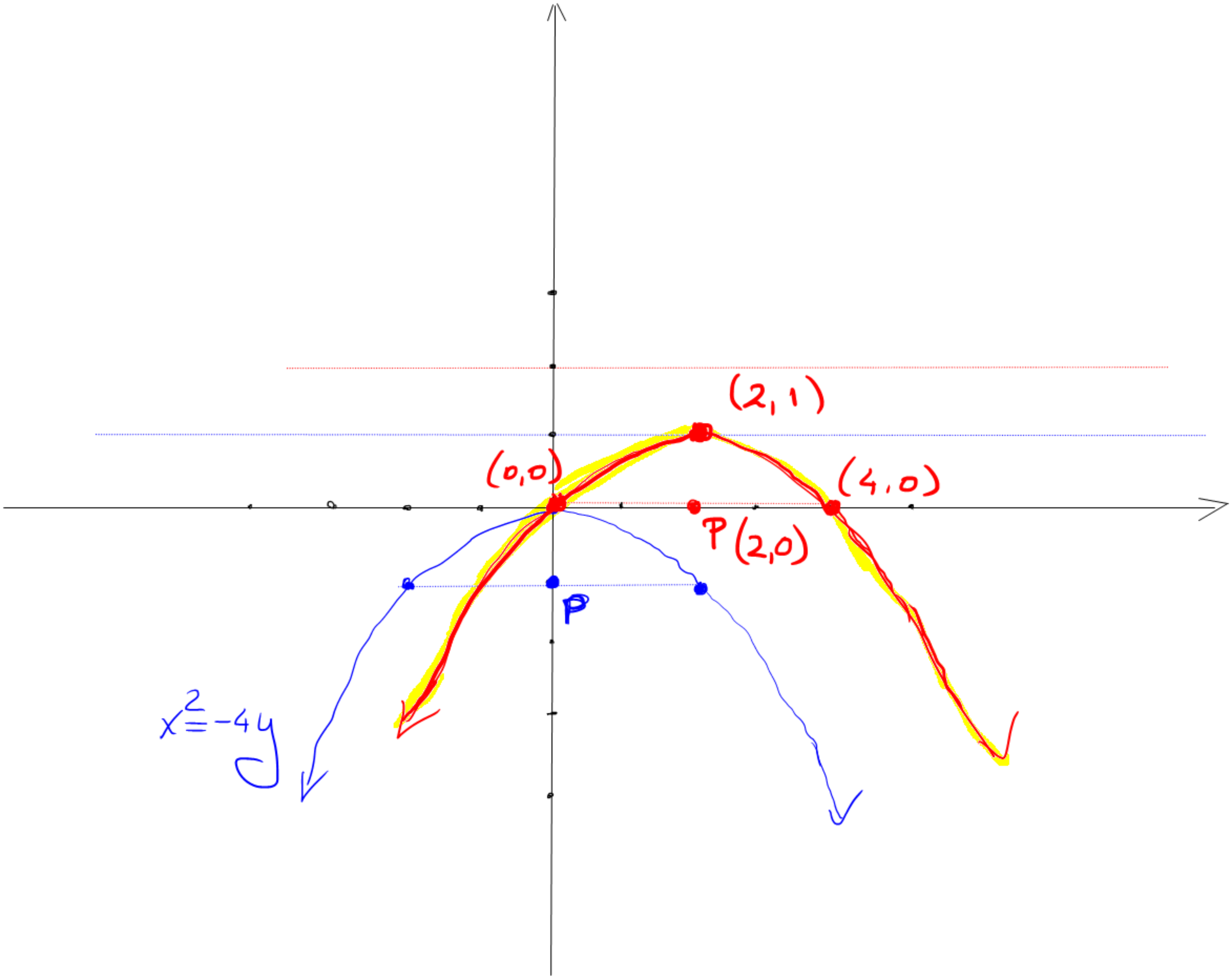
Endpoints of focal chord: $(2,-1)$ and $(-2,-1)$

$-2p$ p $2p$ p

Now graph $(x-2)^2 = -4(y-1)$.

Begin { Basic Parabola $x^2 = -4y$ } $\xrightarrow[\text{1 unit up}]{\text{2 units right}}$ Shifted $(x-2)^2 = -4(y-1)$

- Vertex $(0,0) \longrightarrow (2,1)$
- Focus $(0,-1) \longrightarrow (2,0)$
- Directrix $y = 1 \longrightarrow y = 1+2 = 3$
- Focal cord $= 4 \cdot 1 = 4 \longrightarrow$ same
  - $(2,-1), (-2,-1) \longrightarrow (4,0), (0,0)$



(2,1)
(0,0)
(4,0)
P(2,0)
P
$x^2=-4y$

Our goal: $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$

horizontal vertical

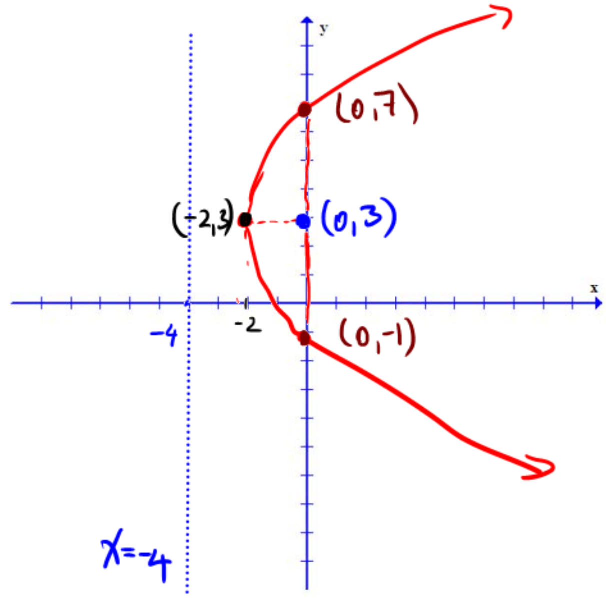
Example 3: Write $y^2 - 6y = 8x + 7$ in standard form and graph it. horizontal

$y^2 - 6y + 9 = 8x + 7 + 9$

$\left(\frac{6}{2}=3\right)^2$ $= 8x + 16$

$(y-3)^2 = \boxed{8}(x+2)$

$8 = 4p \Rightarrow p = 2$



parent

$\boxed{y^2 = 8x}$ $\xrightarrow{\text{shifted}}$ $(y-3)^2 = 8(x+2)$

2 left, 3 up

(0,0) Vertex: $(-2, 3)$ ← apply shiftments

(2,0) Focus: $(0, 3)$ ← apply shiftments

$x = -2$ Directrix: $x = -4$ ← apply shiftments

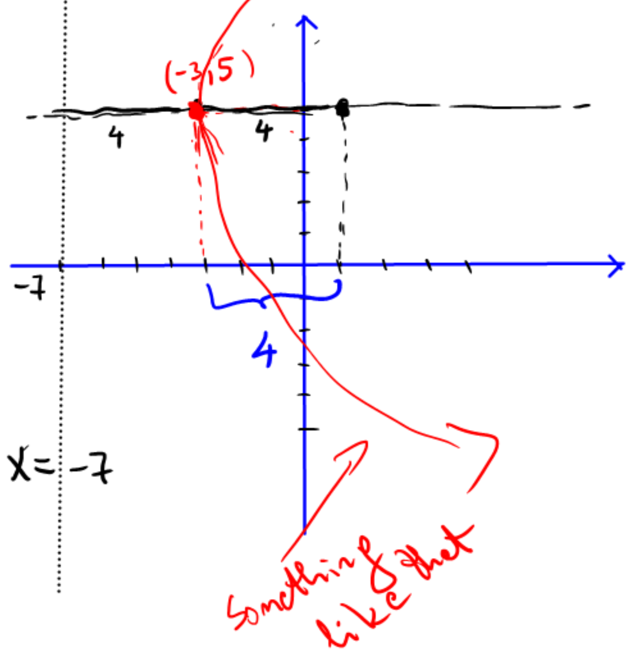
$4 \cdot 2 = 8$ Focal width: $4 \cdot 2 = 8$ same

(2, 4)
(2, −4) Endpoints of focal chord: $\left.\begin{matrix}(0,7)\\(0,-1)\end{matrix}\right\}$ shifted

$p=2$ is always the distance from vertex to focus.

Goal: $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$

Example 4: Suppose you know that the vertex of a parabola is at (-3, 5) and its focus is at (1, 5). Write an equation for the parabola in standard form.



Since vertex and focus are on a horizontal line then we have a horizontal parabola:

$$(y-k)^2 = 4p(x-h)$$

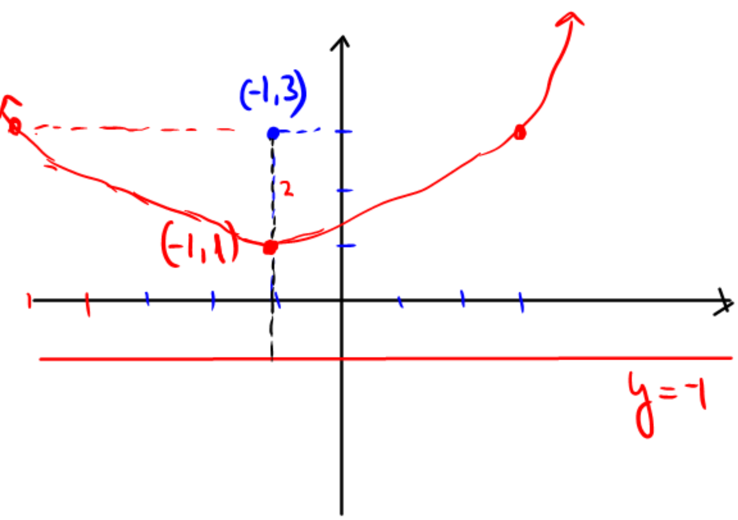
(h,k) = vertex = $(-3,5)$

p = distance btw vertex & focus = 4

$$\Rightarrow (y-5)^2 = 4\cdot 4(x+3) \Rightarrow \boxed{(y-5)^2 = 16(x+3)}$$

exercise

Example 5: Suppose you know that the focus of a parabola is (-1, 3) and the directrix is the line $y = -1$. Write an equation for the parabola in standard form.



Having a horizontal directrix line and a focus above it, gives a vertical upward parabola

$$(x-h)^2 = 4p(y-k)$$

Midpoint of distance from focus to line is the vertex $= (-1,1)$
(h,k)

p = distance of focus and vertex = 2

$$\Rightarrow (x+1)^2 = 4\cdot 2(y-1)$$

$$\boxed{(x+1)^2 = 8(y-1)}$$