Popper 07 <- Bubble

1. State the vertex for the parabola:

$$y^2 - 4x - 8 = 0 \implies y^2 = 4x + 8 \implies y^2 = 4(x+2)$$

Standard Form

$y^2 = 4x$ shifted 2 units left

horizontal

A. $(0,2)$ B. $(0,-2)$ C. $(-2,0)$ D. none

2. State the equation of directrix for this parabola

$$x^2 = 8y = 4 \cdot 2 \cdot y \qquad p = 2 \implies \boxed{y = -2}$$

A. $x=2$ B. $x=-2$ C. $y=2$ D. $y=-2$

3. Given $\frac{x^2}{36} + \frac{y^2}{11} = 1$, find major axis coordinates.

horizontal

$36 = a^2$

$(a,0)$
$(-a,0)$

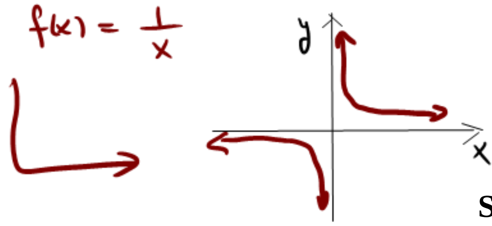
A. $(0,6)$ $(0,-6)$ B. $(6,0)$ $(-6,0)$ C. $(3,0)$ $(-3,0)$ D. $(0,3)$ $(0,-3)$

4. Find the foci points of above ellipse:

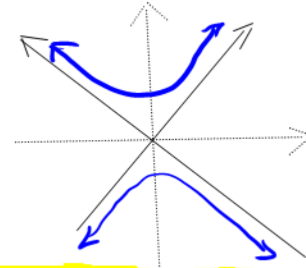
$$c^2 = a^2 - b^2 = 36 - 11 = 25 \implies c = \pm 5$$

A. $(5,0)$ $(-5,0)$ B. $(0,5)$ $(0,-5)$ C. $(\sqrt{11},0)$ $(-\sqrt{11},0)$ D. none.

Think of $f(x) = \frac{1}{x}$



Rotate like $45°$ ↶ $+45°$ Counterclockwise



You get a "Vertical Hyperbola"

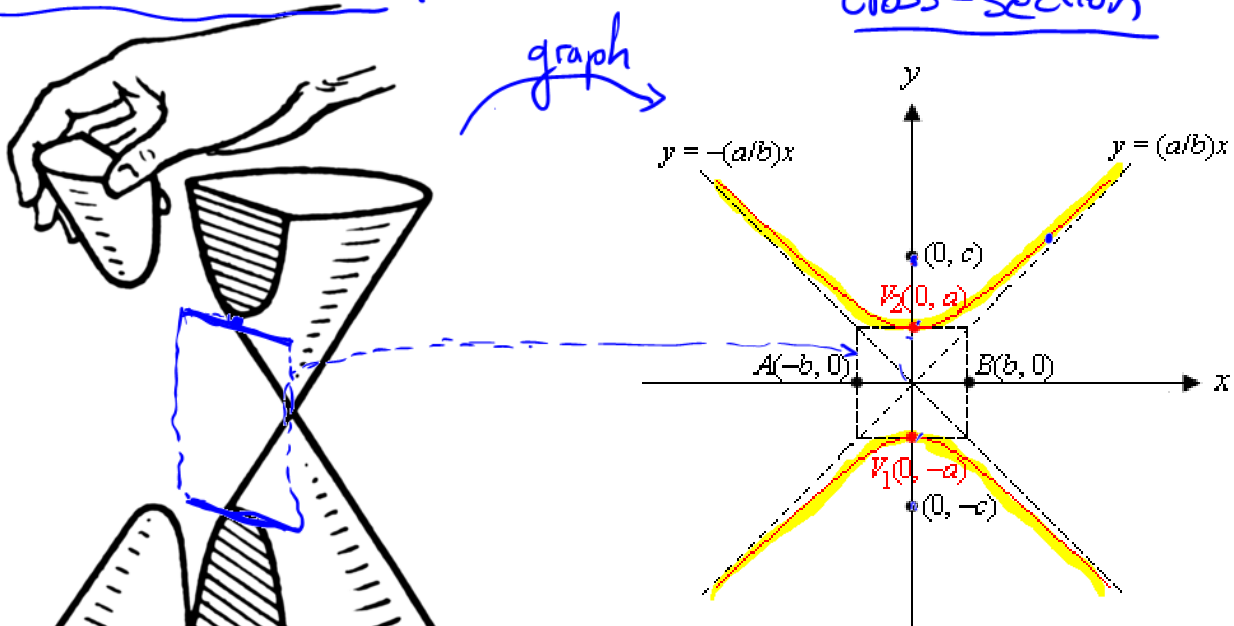
Section 8.3 Hyperbolas

Definition: A *hyperbola* is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a *focus* (plural = *foci*).

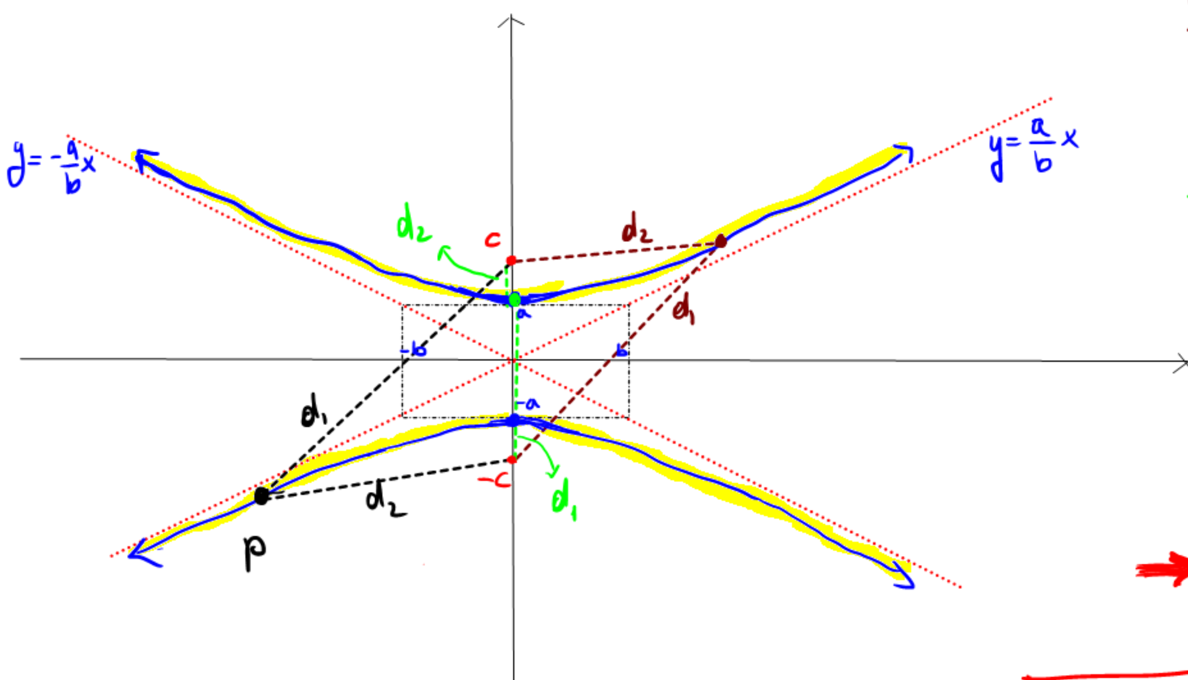
The *focal axis* is the line passing through the foci.

Visualize how it is built!

Cross-section



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$



By definition $d_1 - d_2 = \underbrace{\text{fixed}}_{2a}$

$d_1 - d_2 = \text{fixed}$

$d_1 = c + a$

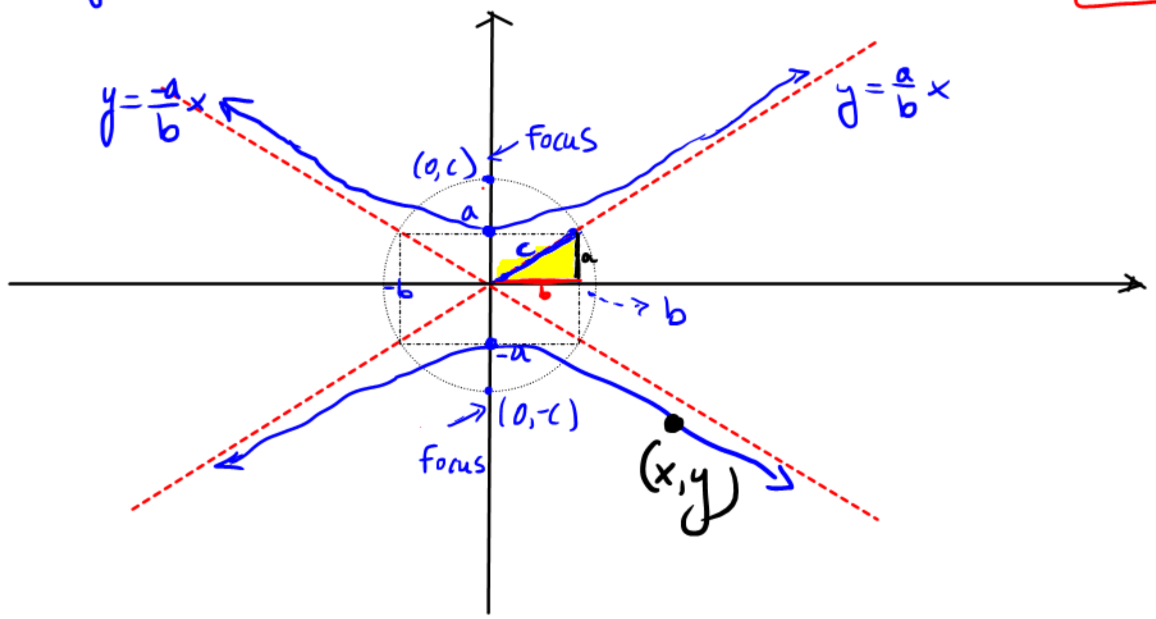
$d_2 = c - a$

$d_1 - d_2 = 2a$

➡ $d_1 - d_2 = 2a$

$\Rightarrow$ By using similar strategy as in parabolas or ellipses, we get $\boxed{c^2 = a^2 + b^2}$

The beauty of hyperbolas is that the foci and the corners of the rectangle are in a circle. This occurs because $\boxed{c^2 = a^2+b^2}$



For any fixed point (x,y) on hyperbola above doing the difference of distances of this point to the foci very similarly as we did for ellipse, but here we have $c^2=a^2+b^2$, we deduce

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

Standard form: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

begins with $\boxed{\dfrac{y^2}{a^2}}$

Basic "vertical" hyperbola:

Equation: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

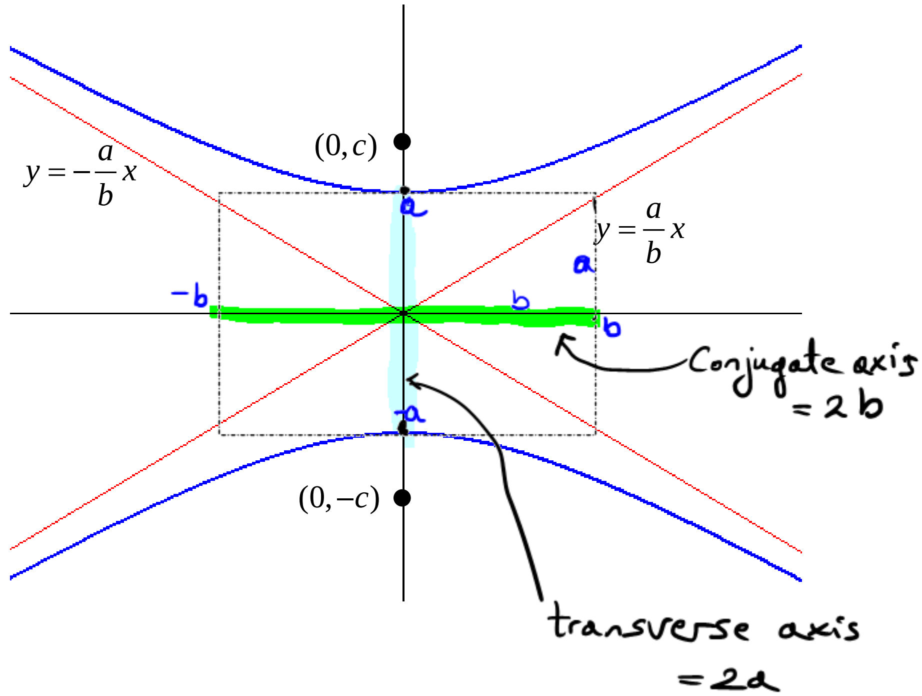
Asymptotes: $y = \pm \dfrac{a}{b} x$

always

Foci: $(0, \pm c)$, where $\boxed{c^2 = a^2 + b^2}$

Vertices: $(0, \pm a)$

Eccentricity: $\dfrac{c}{a}$ (>1)



begins with $\dfrac{x^2}{a^2}$.

Basic "horizontal" hyperbola:

Equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

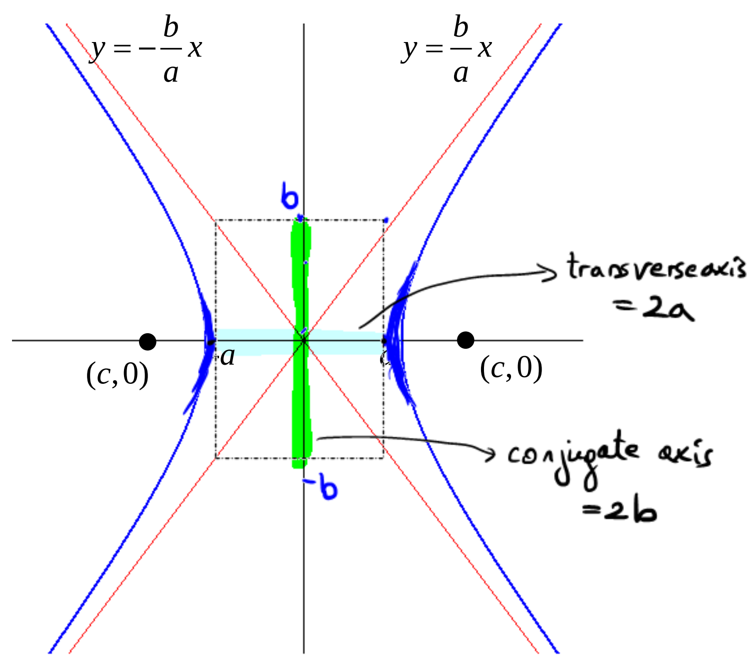
Asymptotes: $y = \pm \dfrac{b}{a} x$

always

Foci: $(\pm c, 0)$, where $\boxed{c^2 = a^2 + b^2}$

Vertices: $(\pm a, 0)$

Eccentricity: $\dfrac{c}{a}$ (>1)



*<u>Note</u>: The **transverse axis** is the line segment joining the two vertices. The **conjugate axis** is the line segment perpendicular to the transverse axis, passing through the center and extending a distance b on either side of the center. (These terms will make more sense after we do the graphing examples.)*

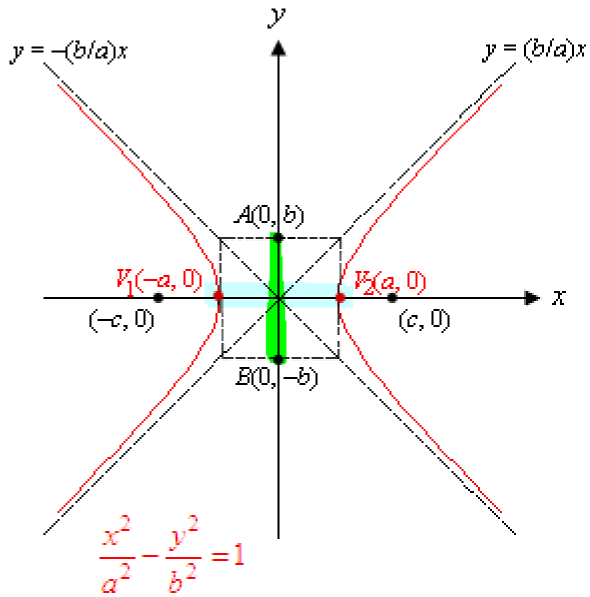
<u>Never forget</u>: Hyperbola curves lie in between two intersecting lines opposite to each other.

Details about conjugate and transverse axis.

$=2b$

The conjugate axis of the hyperbola is the line segment through the center of the hyperbola and perpendicular to the transverse axis with endpoints $(0,-b)$ and $(0,b)$.

$=2a$



Center: $(0,0)$

Foci: $(-c,0)$ and $(c,0)$, where $c^2=a^2+b^2$

Vertices: $V_1(-a,0)$ and $V_2(a,0)$

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

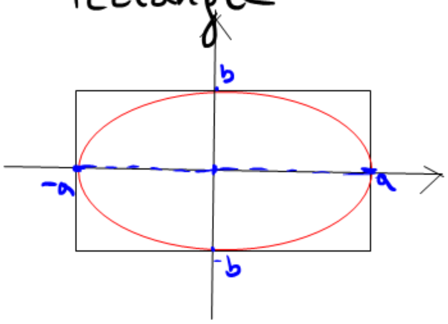
The eccentricity of a hyperbola is given by the formula $e=\dfrac{c}{a}$.

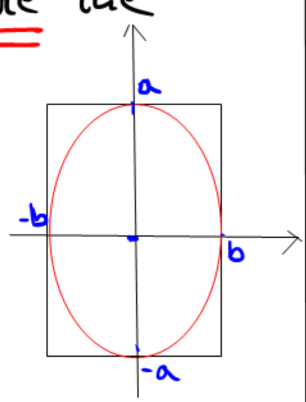
The lines $y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$ are slant asymptotes for the hyperbola since it can be shown that as $|x|$ becomes large, $y\to\pm\dfrac{b}{a}x$.

Ellipses Vs. Hyperbolas

Ellipses

- by definition $d_1 + d_2 = 2a$
- Foci Equation $c^2 = a^2 - b^2$
  (foci are between vertices)
- Horizontal Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
  - Vertices $(a,0)$, $(-a,0)$
- Vertical Ellipse $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
  - Vertices $(0,a)$, $(0,-a)$
- Major axis $= 2a$
  (connects the vertices)
- Minor axis $= 2b$
- No Slant asymptotes
- Ellipse is inside the rectangle



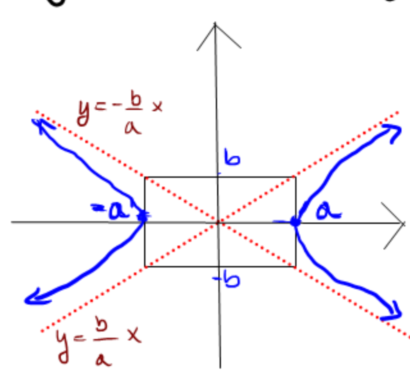


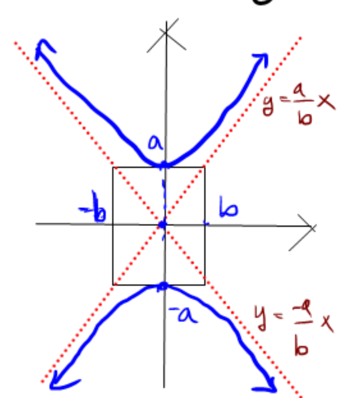
Hyperbolas

- by definition $|d_1 - d_2| = 2a$.
- Foci Equation $c^2 = a^2 + b^2$
  (foci are beyond vertices)
- Horizontal Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
  - Vertices $(a,0)$, $(-a,0)$
- Vertical Hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
  - Vertices $(0,a)$, $(0,-a)$
- Transverse axis $= 2a$
  (connects the vertices)
- Conjugate axis $= 2b$
- Slant asymptotes

  $y = \pm \frac{b}{a} x$ or $y = \pm \frac{a}{b} x$

  Horizontal Vertical
- Hyperbola is beyond the rectangle





To be continued on Wednesday, 2/17.

Graphing hyperbolas: Let's demonstrate by doing example 1.

To graph a hyperbola with center at the origin:

Just know how to begin!!!

- Rearrange into the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.
- Decide if it's a "horizontal" or "vertical" hyperbola.
    - if x^2 comes first, it's horizontal (vertices are on *x*-axis).
    - If y^2 comes first, it's vertical (vertices are on *y*-axis).
- Use the square root of the number under x^2 to determine how far to measure in *x*-direction.
- Use the square root of the number under y^2 to determine how far to measure in *y*-direction.
- Draw a box with these measurements.
- Draw diagonals through the box. These are the asymptotes. Use the dimensions of the box to determine the slope and write the equations of the asymptotes.
- Put the vertices at the edge of the box on the correct axis. Then draw a hyperbola, making sure it approaches the asymptotes smoothly.
- $\boxed{c^2=a^2+b^2}$ where a^2 and b^2 are the denominators.
- The foci are located *c* units from the center, on the same axis as the vertices.

When graphing hyperbolas, you will need to find the orientation, center, values for a, b and c, lengths of transverse and converse axes, vertices, foci, equations of the asymptotes, and eccentricity.

→ let's begin:

Example 1: Find all relevant information and graph $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

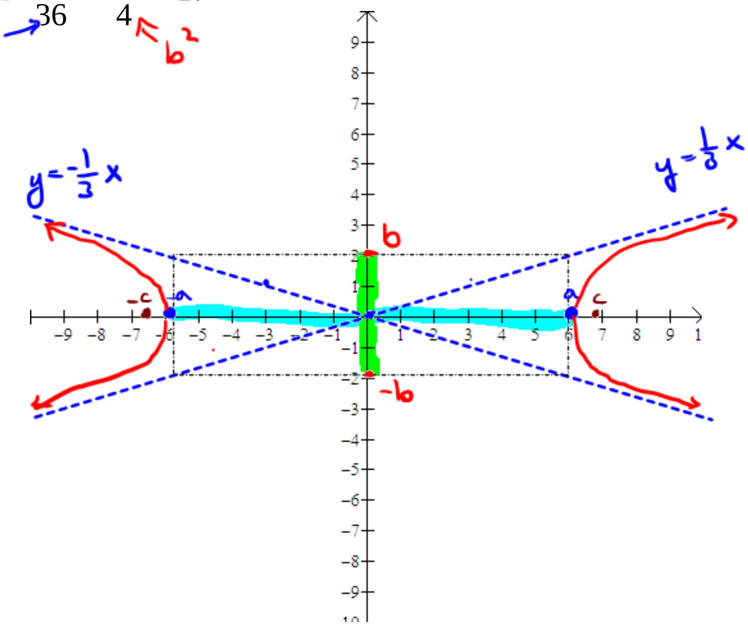
horizontal

$a^2 \to 36$, $4 \leftarrow b^2$

$a^2 = 36 \Rightarrow a = 6$

$b^2 = 4 \Rightarrow b = 2$

$c^2 = a^2 + b^2 = 40 \Rightarrow c = \sqrt{40} = 2\sqrt{10}$



Vertices: $(6,0)$, $(-6,0)$

Foci: $(2\sqrt{10}, 0)$, $(-2\sqrt{10}, 0)$

Eccentricity: $\frac{c}{a} = \frac{2\sqrt{10}}{6} =$

Transverse Axis: = The segment joining vertices

Length of transverse axis:

$2 \cdot a = 2 \cdot 6 = 12$

Conjugate axis: - The segment joining the "b" points.

Length of conjugate axis:

$2 \cdot b = 2 \cdot 2 = 4$

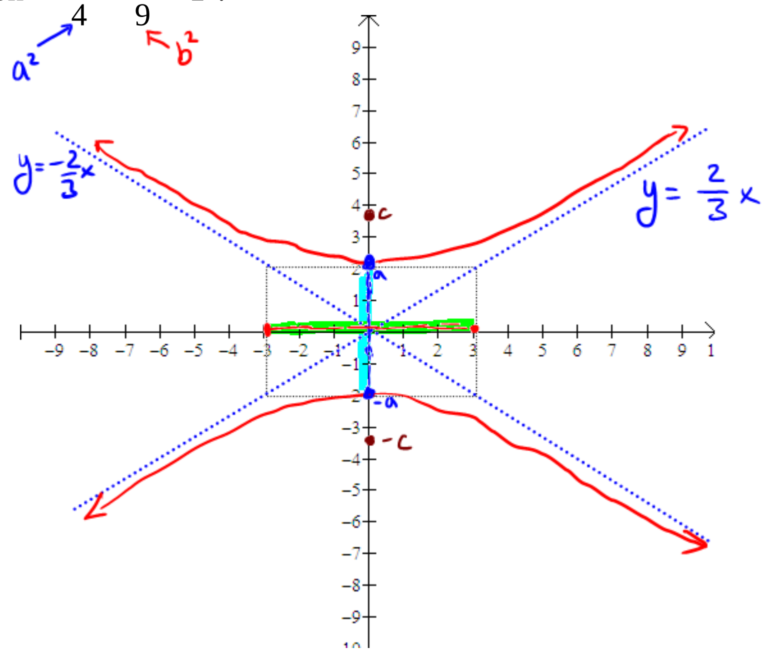
Slant Asymptotes: $y = \frac{b}{a}x = \frac{2}{6}x = \frac{1}{3}x$

$y = -\frac{b}{a}x = -\frac{1}{3}x$

vertical hyperbola with center (0,0)

Example 2: Find all relevant information and graph $\frac{y^2}{4} - \frac{x^2}{9} = 1$.

a^2 b^2



$a^2 = 4 \Rightarrow a = \pm 2$ (vertices)

$b^2 = 9 \Rightarrow b = \pm 3$

$c^2 = a^2 + b^2 = 4 + 9 = 13$

$\Rightarrow c = \pm\sqrt{13}$

Vertices: $(0,2), (0,-2)$

Foci: $(0,\sqrt{13}), (0,-\sqrt{13})$

Eccentricity: $\frac{c}{a} = \frac{\sqrt{13}}{2}$

Transverse Axis: the segment joining vertices

Length of transverse axis: $2a = 2 \cdot 2 = 4$

Conjugate axis: the segment joining "b"

Length of conjugate axis: $2b = 2 \cdot 3 = 6$

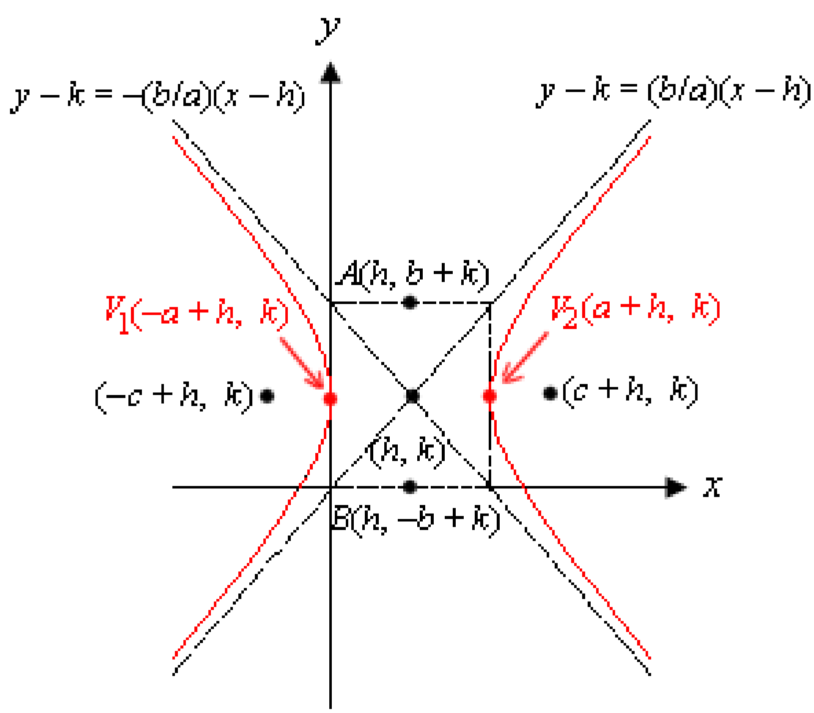
Slant Asymptotes: $y = \pm\frac{a}{b}x = \pm\frac{2}{3}x$

The equation of a hyperbola with center not at the origin: **Center: (h, k)**

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

To graph a hyperbola with center not at the origin:

- Rearrange (complete the square if necessary) to look like
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.
- Start at the center (h,k) and then graph it as before.
- To write down the equations of the asymptotes, start with the equations of the asymptotes for the similar hyperbola with center at the origin. Then replace *x* with $x-h$ and replace *y* with $y-k$.



→ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ shift h units horizontally ⟶ shift k units vertically ⟶ $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

center $(0,0)$ ⟶ center (h,k)

- All the features shift accordingly.

The following list reflects the changes in translating the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$:

Center: The point $(0,0)$ changes to the point (h,k).

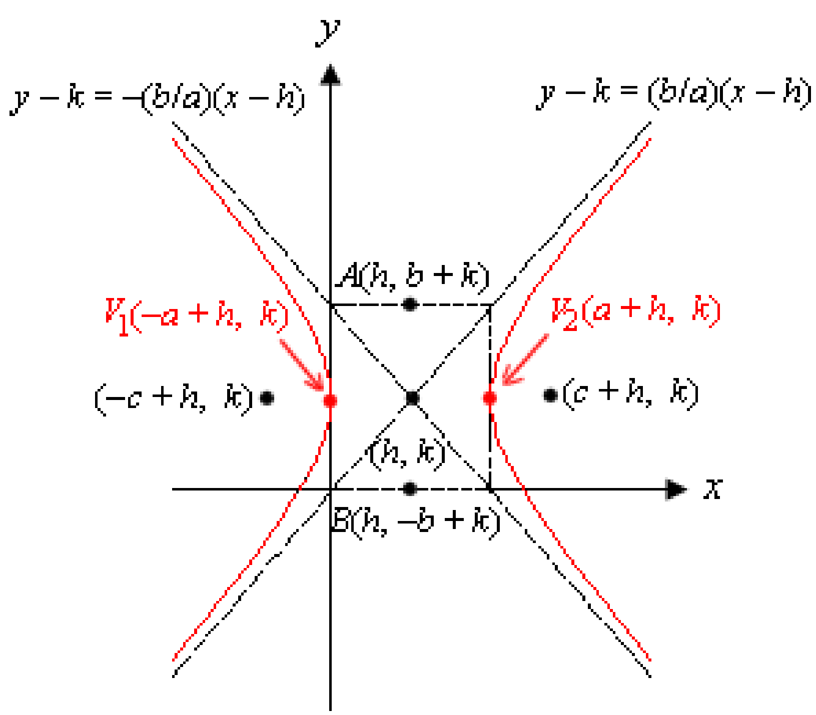
Foci: The foci change from the points $(-c,0)$ and $(c,0)$ to the points $(-c+h,k)$ and $(c+h,k)$, where $c^2=a^2+b^2$.

Vertices: The vertices change from the points $(-a,0)$ and $(a,0)$ to the points $(-a+h,k)$ and $(a+h,k)$.

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

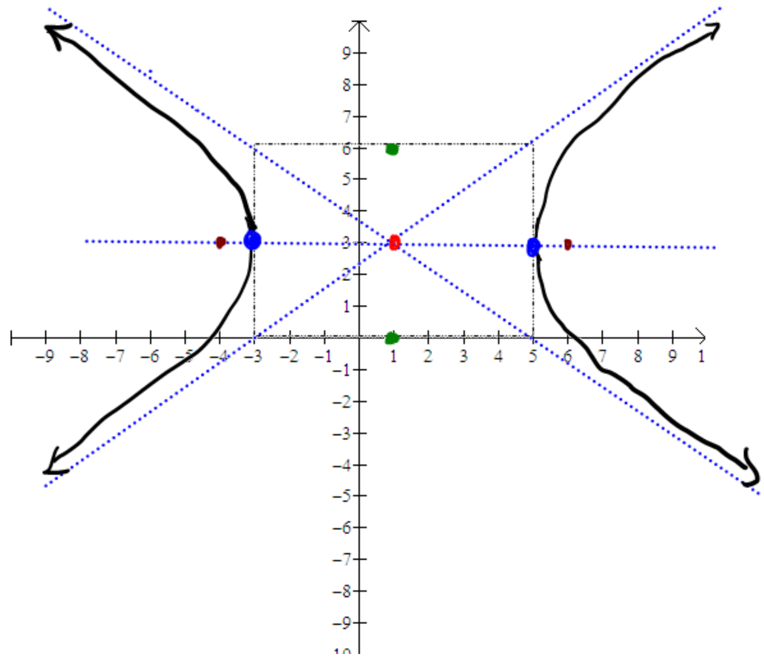
Equations of the Asymptotes: The lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ change to the lines $y-k=\frac{b}{a}(x-h)$ and $y-k=-\frac{b}{a}(x-h)$.



(look at the next page how to bring in standard form:

Example 3: Write the equation in standard form, find all relevant information and graph

$9x^2 - 16y^2 - 18x + 96y = 279$.



Standard form $\rightarrow \dfrac{(x-1)^2}{16} - \dfrac{(y-3)^2}{9} = 1$

$a^2 = 16$, $b^2 = 9$

Horizontal hyperbola with center $(0,0)$	• Horizontal hyperbola shifted 1 right, 3 up center $(1,3)$ ✓
$a^2=16 \Rightarrow a=4$ $(-4,0),(4,0)$	• Vertices shifted $(-3,3)$ ✓, $(5,3)$ ✓
$2a = 2\cdot 4 = 8$	• Transverse axis $= 2a = 8$.
$c^2 = a^2+b^2 = 25$ $c = 5$ $(-5,0),(5,0)$	• Foci Coordinates shifted $(-4,3)$ ✓, $(6,3)$ ✓
$b^2 = 9 \Rightarrow b=3$ $2b = 2\cdot 3 = 6$ $(0,-3),(0,3)$	• Conjugate axis $= 2b = 6$ shifted $(1,0)$ ✓, $(1,6)$ ✓
$y = \frac{b}{a}x = \frac{3}{4}x$ $y = -\frac{b}{a}x = \frac{3}{4}x$	• Slant Asymptotes shifted $y-3 = \frac{3}{4}(x-1)$, $y-3 = -\frac{3}{4}(x-1)$
$e = \frac{c}{a}$	• Eccentricity $e = \frac{c}{a} = \frac{5}{4} = 1.25$

$\Rightarrow$ Draw the rectangle. Then draw diagonals of rectangle and extend. Diagonals are the slant asymptotes

$$9x^2 - 16y^2 - 18x + 96y = 279$$

Likely terms

$$(\underline{9}x^2 - 18x) + (\underline{-16}y^2 + 96y) = 279$$

Factor coefficients and complete square

$$9(x^2 - \underset{\left(\frac{2}{2}=1\right)^2}{2x} + 1) - 16(y^2 - \underset{\left(\frac{6}{2}=3\right)^2}{6y} + 9) = 279 + 9 \cdot 1 - 16 \cdot 9$$

Rewrite

$$9(x-1)^2 - 16(y-3)^2 = 144$$

Divide by 144 both sides

$$\frac{\cancel{9}(x-1)^2}{\cancel{144}\,16} - \frac{\cancel{16}(y-3)^2}{\cancel{144}\,9} = \frac{\cancel{144}}{\cancel{144}}\,1$$

Simplify

$$\boxed{\frac{(x-1)^2}{16} - \frac{(y-3)^2}{9} = 1}$$

Standard form.

This is the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

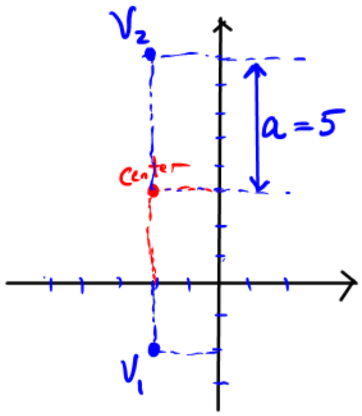
shifted 1 unit right, 3 units up.

Keep in mind : Center, Vertices, Foci are on the same line, always.

Example 4: Write an equation of the hyperbola with center at (-2, 3), one vertex is at (-2, -2) V_1 and eccentricity is 2.

Center gives the shiftment of hyperbola

A quick sketch of problem:

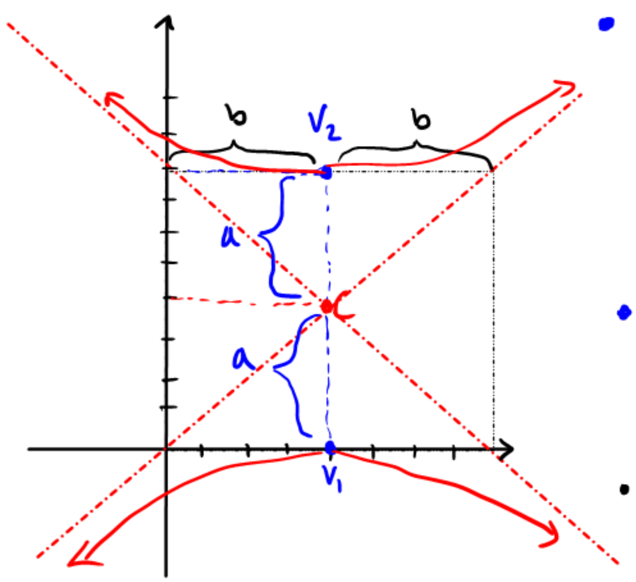


- By picture, it's a vertical hyperbola centered @ (-2,3)
- Vertices are symmetric w.r.t. center, hence $V_2(-2,8)$, and $a=5$. $\Rightarrow a^2=25$
- Need b, $e=\frac{c}{a} \Rightarrow c=e\cdot a=2\cdot 5=10$

$c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2=10^2-5^2=75$

$$\Rightarrow \boxed{\frac{(y-3)^2}{25}-\frac{(x+2)^2}{75}=1}$$

Exercise **Example 5:** Write an equation of the hyperbola if the vertices are $\underbrace{(4, 0)}_{V_1}$ and $\underbrace{(4, 8)}_{V_2}$ and the asymptotes have slopes ± 1.



- Vertices lined vertically $\Rightarrow$ Vertical hyperbola.

Center is the midpoint of transverse axis.

$\Rightarrow$ $\underbrace{C=(4,4)}$ shifts of hyperbola.

- $a=$ half of transverse length $=8/2=4$.

$a^2=16$.

- Being vertical, slant asymptotes are $y=\pm\frac{a}{b}x$

$\Rightarrow \frac{a}{b}=1 \Rightarrow a=b \Rightarrow b^2=16$

$$\Rightarrow \boxed{\frac{(y-4)^2}{16}-\frac{(x-4)^2}{16}=1}$$

Popper 08 --- Bubble correctly!

Identify the following:

A) Circle **B) Ellipse** **C) Parabola** **D) Hyperbola** **E) None**

D **Question #1** : $\frac{(x+4)^2}{4} - \frac{(y-1)^2}{9} = 1$ ⇒ hyperbola shifted 4 units left, 1 unit up.

B **Question #2** : $x^2 + 2y^2 - 4x + y = 9$ ⇒ ellipse, both square terms are positive with different coefficients

$(x-2)^2 + 2(y-0.5)^2 = 13.5$

$\Rightarrow \frac{(x-2)^2}{13.5} + \frac{(y-0.5)^2}{6.75} = 1$

D **Question #3** : $x^2 - 4x - y^2 + 6y = 10$ ⇒ hyperbola, because square terms have opposite signs

$(x-2)^2 - (y-3)^2 = 23$

$\frac{(x-2)^2}{23} - \frac{(y-3)^2}{23} = 1$

C **Question #4** : $x^2 - 4x + 12y = 9$ ⇒ parabola, just one square term showing

A **Question #5** : $9x^2 + 9y^2 - 4x + 18y = 9$ ⇒ circle, both positive square terms and same coefficient

$9(x^2 + y^2 - \frac{4}{9}x + 2y) = 9$

$x^2 - \frac{4}{9}x + y^2 + 2y = 1$

$(x - \frac{2}{9})^2 + (y+1)^2 = 1 + \frac{4}{81} + 1 = \frac{166}{81}$