Math 1330 - Section 1.4
Combining Functions

– A new function

We can combine functions in any of five ways. Four of these are the familiar arithmetic operations; addition, subtraction, multiplication and division, and are very intuitive. The fifth type of combining functions is called composition of functions. In all cases, we'll be interested in combining the functions and in finding the domain of the combined function.

Suppose we have two functions, $f(x)$ with domain *A* and $g(x)$ with domain *B*.

- $f(x)=\sqrt{x}$
  dom $f = [0,\infty)$
- $g(x)=\dfrac{1}{x-5}$
  dom $g=(-\infty,5)\cup(5,\infty)$

Sum: $(f+g)(x)=f(x)+g(x)$

$= \sqrt{x}+\dfrac{1}{x-5}$, both: $x \geq 0,\ x \neq 5$

dom $(f+g) = [0,5)\cup(5,\infty)$

Difference: $(f-g)(x)=f(x)-g(x)$

$= \sqrt{x}-\dfrac{1}{x-5}$, both: $x \geq 0,\ x \neq 5$

dom $(f-g) = [0,5)\cup(5,\infty)$

Product: $(fg)(x)=f(x)g(x)$

$= \sqrt{x}\cdot\dfrac{1}{x-5} = \dfrac{\sqrt{x}}{x-5}$, both: $x \geq 0,\ x \neq 5$

dom $(f\cdot g) = [0,5)\cup(5,\infty)$

Quotient: $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$

$= \dfrac{\sqrt{x}}{\frac{1}{x-5}} = \sqrt{x}\cdot(x-5)$, dom$\left(\dfrac{f}{g}\right) = [0,5)\cup(5,\infty)$

$x \geq 0,\ x \neq 5$ both and $g(x) \neq 0$.

The final way of combining functions is called **composition of functions**.

$f(x)=\sqrt{x}$

$g(x)=\dfrac{1}{x-5}$

$(f \circ g)(x)=f(g(x)) = \sqrt{g(x)} = \sqrt{\dfrac{1}{x-5}} \Rightarrow$ $x \neq 5$ (inner), $x-5>0 \Rightarrow x>5$

inner

The domain of $f \circ g$ is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

To find domain of $f\circ g$: Find domain of result and domain of inner function.

$\Rightarrow$ dom $(f\circ g) = (5,\infty)$

- For $f(x) = \sqrt{x}$, $\operatorname{dom} f = [0, \infty)$, $x \geq 0$

  $g(x) = \dfrac{1}{x-5}$, $\operatorname{dom} g = (-\infty, 5) \cup (5, \infty)$

  $x \neq 5$

$$(g \circ f)(x) = g\big(f(x)\big) = \frac{1}{f(x) - 5} = \frac{1}{\sqrt{x} - 5}$$

Domain : I. of course

$\sqrt{x} - 5 \neq 0$

$$\left\{ \begin{array}{l} \sqrt{x} - 5 = 0 \\ (\sqrt{x} = 5)^2 \\ x = 25 \end{array} \right. \Longrightarrow x \neq 25$$

by result,

II. and

$x \geq 0$ from inner function f.

$\operatorname{dom} g \circ f$: $x \neq 25$, $x \geq 0$

$= [0, 25) \cup (25, \infty)$. $\square$

Example 1: Suppose $f(x) = 2x+3$ and $g(x) = 4x-8$.

Find $f+g$, $f-g$, fg, $\dfrac{f}{g}$, $f \circ g$ and $g \circ f$.

$\text{dom} f = (\infty, \infty)$

$\text{dom} g = (-\infty, \infty)$

- $(f+g)(x) = (2x+3)+(4x-8)$
  $= 6x-5$
  $\Rightarrow \text{dom}(f+g) = (-\infty, \infty)$

- $(f-g)(x) = (2x+3)-(4x-8)$
  $= -2x+11$
  $\Rightarrow \text{dom}(f-g) = (-\infty, \infty)$

- $(fg)(x) = (2x+3)(4x-8)$
  $= 16x^2-4x-24$
  $\Rightarrow \text{dom}(f \cdot g) = (-\infty, \infty)$

- $\left(\dfrac{f}{g}\right)(x) = \dfrac{2x+3}{4x-8}$
  $\Rightarrow \text{dom}\left(\dfrac{f}{g}\right) = (-\infty, 2) \cup (2, \infty)$

  $g(x) \neq 0$, $4x-8=0$
  $x \neq 2$

  $x \neq 2$

- $(f \circ g)(x) = f(g(x))$
  $= 2 \cdot g(x) + 3$
  $= 2 \cdot (4x-8) + 3$
  $= 8x - 13$
  $\Rightarrow \text{dom}(f \circ g) = (-\infty, \infty)$

- $(g \circ f)(x) = g(f(x))$
  $= 4 \cdot f(x) - 8$
  $= 4(2x+3) - 8$
  $= 8x + 4$
  $\Rightarrow \text{dom}(g \circ f) = (-\infty, \infty)$

Example 2: Suppose $f(x)=\dfrac{2}{x-1}$ and $g(x)=\dfrac{x}{2x+4}$.

Find:

a) $(f\circ g)(2) = f(g(2)) = f(\frac{1}{4})$

$= \dfrac{2}{\frac{1}{4}-1\cdot\frac{4}{4}} = \dfrac{2}{\frac{-3}{4}} = \dfrac{-8}{3}$

$g(2) = \dfrac{2}{2\cdot 2+4} = \dfrac{2}{8} = \dfrac{1}{4}$

$f(3) = \dfrac{2}{3-1} = \dfrac{2}{2} = 1$

b) $g(f(3))$

$= g(1) = \dfrac{1}{2\cdot 1+4} = \dfrac{1}{6}$

c) $(f\circ g)(x)$

$= f(g(x)) = \dfrac{2}{g(x)-1} = \dfrac{2}{\frac{x}{2x+4}-1} = \dfrac{2}{\frac{x}{2x+4}-\frac{2x+4}{2x+4}}$

$= \dfrac{2}{\frac{-x-4}{2x+4}} = \dfrac{2(2x+4)}{-x-4} = \dfrac{4x+8}{-(x+4)} = \underbrace{-\dfrac{4x+8}{x+4}}$.

d) Domain of $(f\circ g)(x)$: I. result $= \dfrac{-(4x+8)}{x+4}$, $x+4\neq 0 \Rightarrow \boxed{x\neq -4}$

II. dom of g: $\dfrac{x}{2x+4}$, $2x+4\neq 0 \Rightarrow \boxed{x\neq -2}$

$2x=-4$

$x=-2$

both

$$\boxed{\text{dom}(f\circ g) = (-\infty,-4)\cup(-4,-2)\cup(-2,\infty)}$$

exercise e) Domain of $(g\circ f)(x)$:

$(g\circ f)(x) = \ldots = \dfrac{1}{2x} \Rightarrow$ I. $x\neq 0$ (result) II: $f(x) \Rightarrow x\neq 1$ $\Rightarrow$ dom $(g\circ f)$ =

$(-\infty,0)\cup(0,1)\cup(1,\infty)$

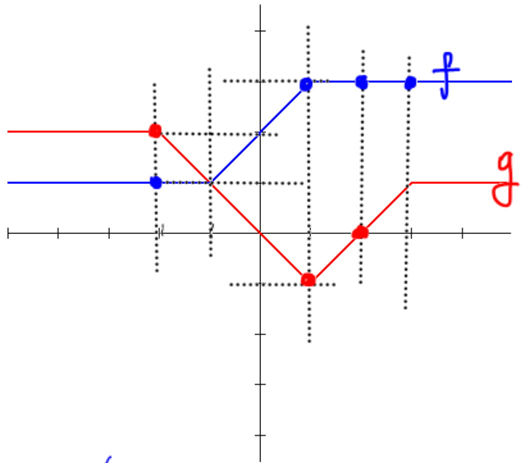
f) $(fg)(0)$

$= f(0)\cdot g(0) = \left(\dfrac{2}{0-1}\right)\cdot\left(\dfrac{0}{2\cdot 0+4}\right) = 0.$

Note: $(fg)(x)$ **and** $f(g(x))$ **have different meanings!!!! Be careful about the notation.**

Example 3: In the graph below, the function graphed in blue is $f(x)$ and the function graphed in red is $g(x)$. Find each quantity.

$g(-2) = 2$
$g(1) = -1$
$g(2) = 0$



$f(-2) = 1$
$f(1) = 3$
$f(2) = 3$
$f(3) = 3$

a. $(f+g)(-2) = f(-2) + g(-2) = 1 + 2 = 3$

b) $(fg)(-2) = f(-2) \cdot g(-2) = 1 \cdot 2 = 2$

c) $f(\underbrace{g(-2)}_{2}) = f(2) = 3$

d. $g(\underbrace{f(-2)}_{1}) = g(1) = -1$

e. $f(\underbrace{g(3)}_{1}) = f(1) = 3$

f. $g(f(0)) = g(2) = 0$

g. $f(f(2)) = f(3) = 3$