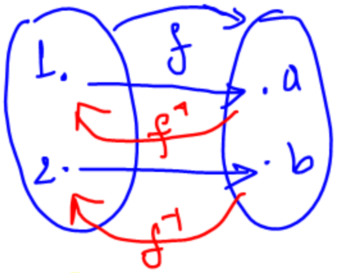
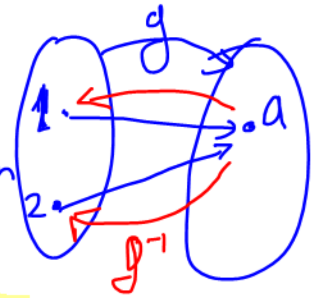
f function



f^{-1} function

BUT

g function



g^{-1} not a function anymore

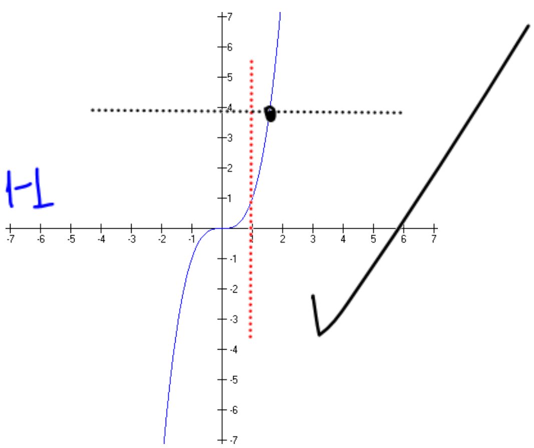
Math 1330 - Section 1.5
Inverse Functions

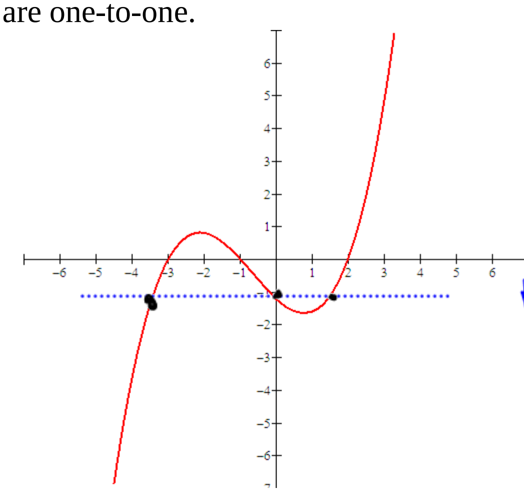
A function is **one-to-one** if no two elements in the domain have the same image.

The Horizontal Line Test: A function is one-to-one if any horizontal line intersects the graph of the function in no more than one point. (H.L.T.)

Example 1: Determine if the functions graphed are one-to-one.

1-1



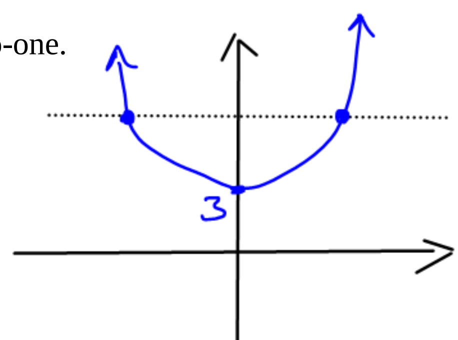


No, H.L.T. fails.

Example 2: Determine if $f(x) = x^2 + 3$ is one-to-one.

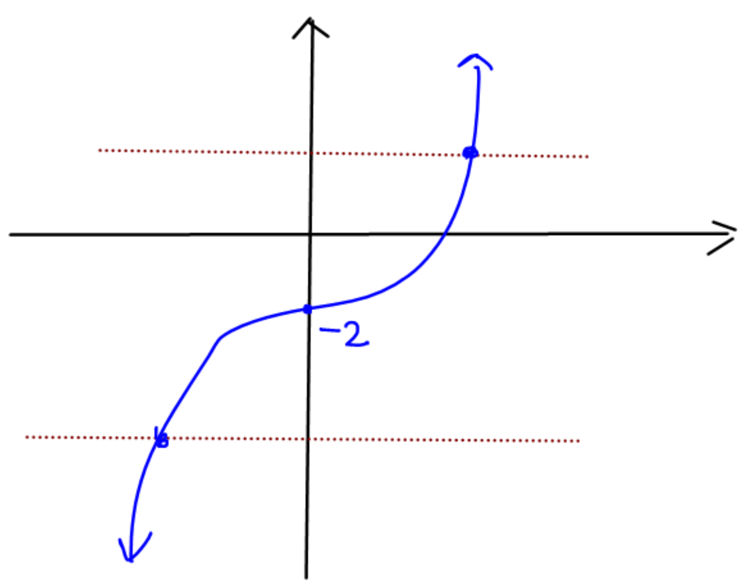
$x = 1 \Rightarrow f(1) = 1^2 + 3 = 4$

$x = -1 \Rightarrow f(-1) = (-1)^2 + 3 = 4$



H.L.T. fails, not 1-1.

Example 3: Determine if $f(x) = x^3 - 2$ is one-to-one.

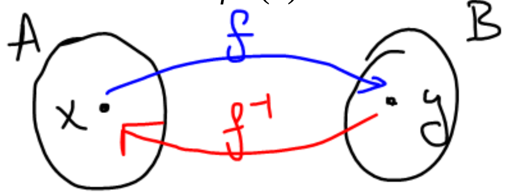


Yes, f is 1-1.

Only 1-1 functions have inverses!!!

If a function is one-to-one then there is an associated function called "the inverse". $= f^{-1} = f$ inverse

The **inverse function** of a one-to-one function is a function $f^{-1}(x)$ such that $\boxed{\left(f \circ f^{-1}\right) = \left(f^{-1} \circ f\right) = x.}$



$f(x) = y$

$f^{-1}(y) = x$

$f^{-1}(f(x)) = x$

* Note: $f^{-1}(x) \neq \dfrac{1}{f(x)}$

To determine if two functions are inverses of one another, you need to compose the functions in both orders. Your result should be *x* in both cases. That is, given two functions *f* and *g*, the functions are inverses of one another if and one if $f(g(x)) = g(f(x)) = x$.

Example 4: Determine if $f(x) = 5x - 2$ and $g(x) = \dfrac{x-2}{5}$ are inverses of one another.

Look at $(f \circ g)(x) = x$???

$\Rightarrow (f \circ g)(x) = f(g(x)) = 5 \cdot g(x) - 2 = \cancel{5} \cdot \dfrac{x-2}{\cancel{5}} - 2$

$= x - 2 - 2 = x - 4 \neq x.$

Hence, f and g are not inverses!

Note: The inverse function reverses what the function did. Therefore, the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Example5: If $f(-1) = 2$, $f^{-1}(-1) = 0$ and $f(2) = 5$, find $f(0)$ and $f^{-1}(5)$.

$f^{-1}(-1) = 0 \Rightarrow f(0) = 1$

$f(2) = 5 \Rightarrow f^{-1}(5) = 2$

Example 6: Find the linear function f if $f^{-1}(4) = 0$ and $f^{-1}(2) = 1$.

Recall:

$(x_1, y_1), (x_2, y_2)$

slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$

equation:

$y - y_1 = m(x - x_1)$

$f^{-1}(4) = 0 \Rightarrow f(0) = 4 \Rightarrow (0, 4)$

$f^{-1}(2) = 1 \Rightarrow f(1) = 2 \Rightarrow (1, 2)$

$m = \dfrac{4-2}{0-1} = -2 \Rightarrow y - 4 = -2(x - 0)$

$\Rightarrow \boxed{y = -2x + 4}$

extra example:

Show $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$ are inverses.

$\boxed{\text{YES}}$

Check: $(f \circ g)(x) = x$?

$$(f \circ g)(x) = f\big(g(x)\big) = \big(g(x)\big)^3 + 1$$

$$= \left(\sqrt[3]{x-1}\right)^3 + 1$$

$$= x - 1 + 1 = x$$ ✓

Now, check: $(g \circ f)(x) = x$?

$$(g \circ f)(x) = g\big(f(x)\big) = \sqrt[3]{f(x) - 1}$$

$$= \sqrt[3]{x^3 + 1 - 1} = \sqrt[3]{x^3} = x$$ ✓

You need to be able to find the inverse of a function. Follow this procedure to find an inverse function:

Steps

1. Rewrite the function as $y = f(x)$.
2. Interchange x and y.
3. Solve the equation you wrote in step 2 for y.
4. Rewrite the inverse using inverse notation, $f^{-1}(x)$.

Inverse is some function y s.t. $\underbrace{f(y) = x}_{\text{composition}}$

Example 7: You know that $f(x) = 4x - 7$ is a one-to-one function. Find its inverse.

Rewrite ① $y = 4x - 7$

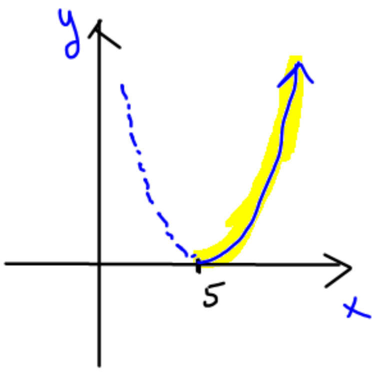
Exchange ② $x = 4y - 7$

Solve ③ $x = 4y - 7$

$\frac{x+7}{4} = \frac{4}{4}y \Rightarrow y = \frac{x+7}{4}$

Answer ④ $\boxed{f^{-1}(x) = \frac{x+7}{4}}$

Example 8: Determine if $f(x) = (x-5)^2$, $x \geq 5$ is a one-to-one function. If it is, find its inverse.



Half of parabola is 1–1 : $[5, \infty) \xrightarrow{f} [0, \infty)$

① $y = (x-5)^2$

② $x = (y-5)^2$

③ Solve for y

$\sqrt{(y-5)^2} = \sqrt{x}$

$y - 5 = +\sqrt{x}$

(range f positive)

$y = \sqrt{x} + 5$.

④ $\boxed{f^{-1}(x) = \sqrt{x} + 5}$

Example 9: $f(x) = \dfrac{1+x}{2-x}$ is a one-to-one function. Find its inverse.

① $y = \frac{1+x}{2-x} \Rightarrow$ ② $x = \frac{1+y}{2-y} \Rightarrow$ ③ Solve for y

$\frac{x}{1} = \frac{1+y}{2-y} \Leftrightarrow$

$1+y = x(2-y)$

$1+y = 2x - xy$

$xy + y = 2x - 1$

$y(1+x) = 2x - 1$

$y = \frac{2x-1}{x+1}$

$\Rightarrow$ ④ $\boxed{f^{-1}(x) = \frac{2x-1}{x+1}}$

(Extra) Example 10: Find the inverse of the function $f(x) = 5 + \sqrt{4x+1}$.

- It is 1-1 ✓

- Domain of f : $4x+1 \geq 0$
  $x \geq -\frac{1}{4}$

f^{-1}

- Range of f : $[5, \infty)$

Find inverse:

① $y = 5 + \sqrt{4x+1}$

② $x = 5 + \sqrt{4y+1}$

③ Solve for y:

$$5 + \sqrt{4y+1} = x$$

$$\sqrt{4y+1} = x - 5 \leftarrow \text{Square both sides}$$

$$4y + 1 = (x-5)^2$$

$$y = \frac{(x-5)^2 - 1}{4} = \frac{1}{4}(x-5)^2 - \frac{1}{4}$$

④ $\boxed{f^{-1}(x) = \frac{1}{4}(x-5)^2 - \frac{1}{4}, \quad [5, \infty)}$