Math 1330 - Section 2.2
Polynomial Functions

Our objectives in working with polynomial functions will be, first, to gather information about the graph of the function and, second, to use that information to generate a reasonably good graph without plotting a lot of points. In later examples, we'll use information given to us about the graph of a function to write its equation.

A **polynomial function** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$, $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number.

✓ $f(x) = 2x^3 + 3x^2 - 5$

✓ $f(x) = 2x^{10} + \sqrt{3}\,x^7 - x + 0$

✗ $f(x) = x^4 + \sqrt{x}$

The number a_n is called the **leading coefficient**. The degree of the polynomial function is n.

$f(x) = 5x^3 - 3x^4 + 2x - 5$ → $f(0) = -5$

$P(0) = a_0$ and this number is called the **constant coefficient**.

Highest exponent gives the degree of polynomial → if even, then "even-degree" polynomial; if odd, then "odd-degree" polynomial

Example: The graph of $f(x) = x(x-2)^3(x+1)^2$ is given:

⟹ find degree: Need "leading term"

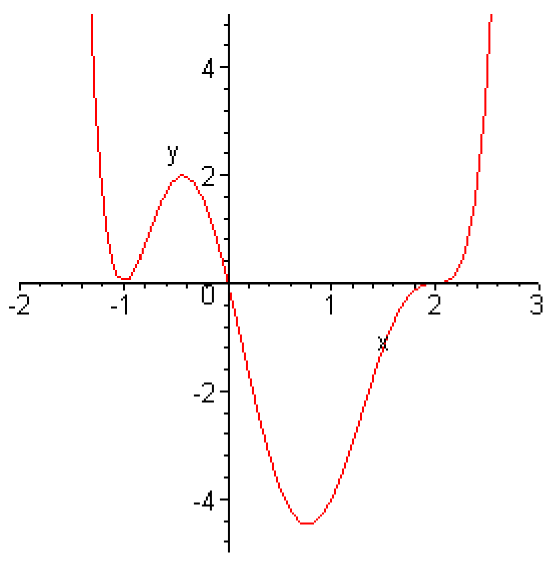
$f(x) = x(x-2)^3(x+1)^2$

$= x(x^3 + \ldots)(x^2 + \ldots)$

⟹ leading term: $x \cdot x^3 \cdot x^2 = x^{1+3+2} = x^6$ (degree)

It is a 6^{th} degree polynomial

i.e. even degree polynomial



⟹ leading coefficient $= +1$ i.e. positive

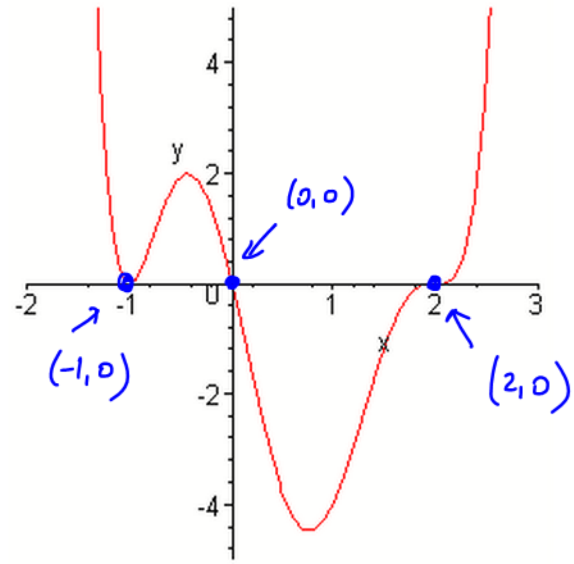
⟹ end behaviour is upward ↑ . . . ↗

What else we can find:

$f(x) = x(x-2)^3(x+1)^2$

- x-intercept(s) : $f(x)=0$

$\Rightarrow$ $x=0$ (repeated once)
behaves like a "line"
at $(0,0)$



$\Rightarrow$ $x-2=0$ i.e. $x=2$ (repeated 3 times)
behaves like an "cube" shape at $(2,0)$

$\Rightarrow$ $x+1=0$ i.e. $x=-1$ (repeated 2 times)
behaves like a "parabola" shape at $(-1,0)$

Leading term of a polynomial is very important. It determines:

- the degree of polynomial (value of highest exponent)
- the graphic behaviour of polynomial (value of coefficient determines whether is upward or downward at the endpoints of graph).

ex a) $f(x) = \boxed{-3x^2} + 5x - 4$

leading term, degree 2, downward.

b) $f(x) = \boxed{5x^3} + 4x^2 - 5x + 10$

leading term, degree 3, like x^3.

Basic even power functions – parabola shape

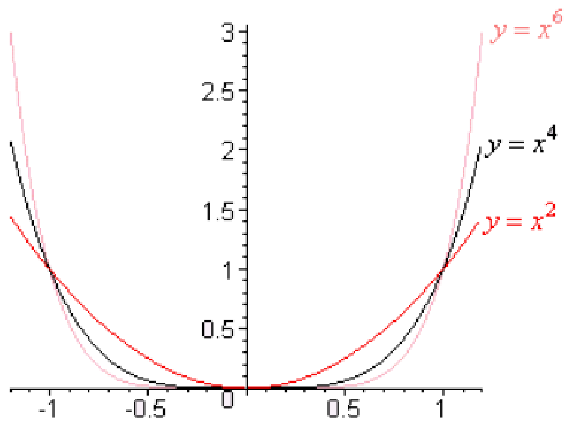
Basic odd power functions – cube shape

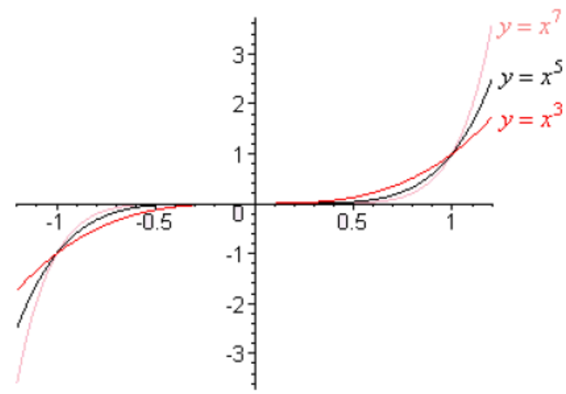
$y = x^2$, $y = x^4$, $y = x^6$, $y = x^8$, $y = x^{10}$, ⋮

$y = x^3$, $y = x^5$, $y = x^7$, $y = x^1$, $y = x^4$, ⋮

From college algebra, you should be familiar with the graphs of $f(x) = x^2$ and $g(x) = x^3$.

The graph of $f(x) = x^n$, $n > 0$, n is even, will resemble the graph of $f(x) = x^2$, and the graph of $f(x) = x^n$, $n > 0$, n is odd, will resemble the graph of $f(x) = x^3$.



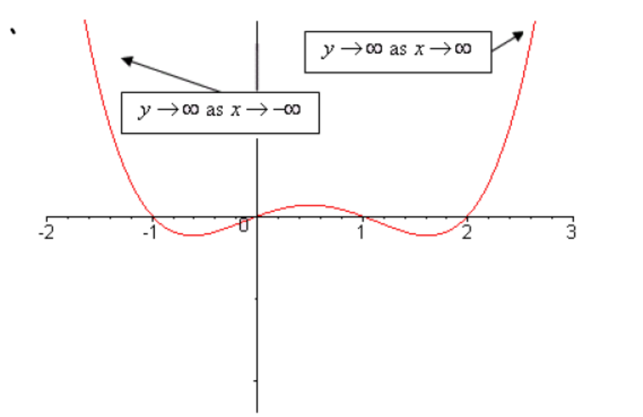


Think of $f(x) = \boxed{x^4} + 10x^2 - 3x + 5$ → as x gets very BIG, f(x) begins to behave just like "x^4"

Next, you will need to be able to describe the **end behavior** of a function.

If the degree of the function is even and $a_n > 0$, then the end behavior of the function is
Degree: Even, Coefficient: +

***n* even and $a_n > 0$ (even degree and leading coefficient positive)**



$f(x) = 3x^4$ + the rest

- degree f = 4 – even
- leading coefficient = ⊕

→ end behaviour ↖ . . . ↗

(think of "x^2")

If the degree of the function is even and $a_n < 0$, then the end behavior of the function is

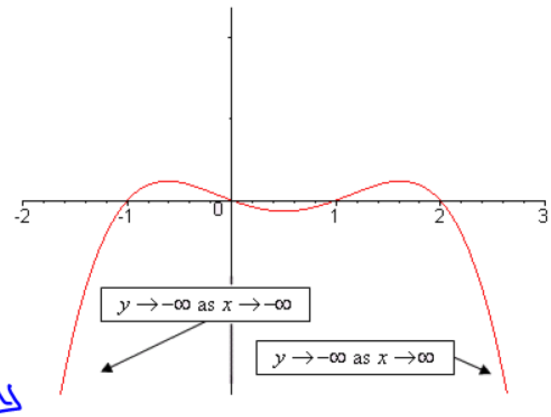
Degree: Even, Coefficient: -

$f(x) = -3x^4 + \text{the rest}$

- degree f = 4 - even
- leading coefficient = $-$

→ end behaviour ↙ . . . ↘

n even and $a_n < 0$ (even degree and leading coefficient negative)



(think of "$-x^2$")

If the degree of the function is odd and $a_n > 0$, then the end behavior of the function is

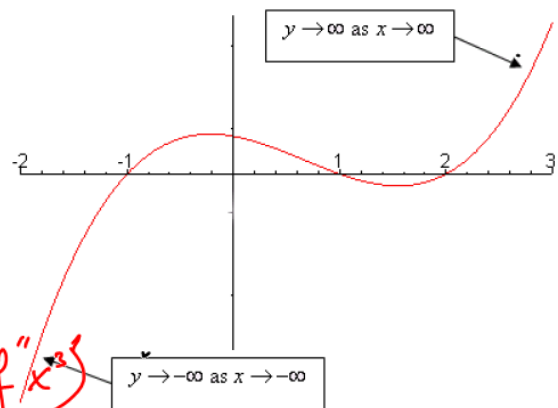
Degree: Odd, Coefficient: +

$f(x) = 5x^5 + \text{the rest}$

- degree f = 5 odd
- leading coefficient = $+$

→ end behaviour ↙ . . . ↗

n odd and $a_n > 0$ (odd degree and leading coefficient positive)



(think of "x^3")

If the degree of the function is odd and $a_n < 0$, then the end behavior of the function is

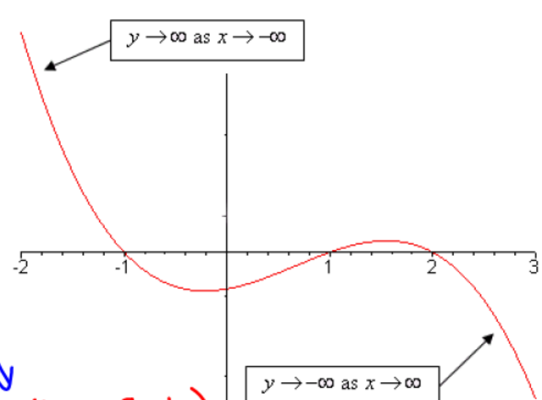
Degree: Odd, Coefficient: -

$f(x) = -5x^5 + \text{the rest}$

- degree f = 5 - odd
- leading coefficient = $-$

→ end behaviour ↖ . . . ↘

n odd and $a_n < 0$ (odd degree and leading coefficient negati



(think of "$-x^3$")

↙ ↗ Rising left to right ↖ ↘ Falling left to right

↖ ↗ Both ends up ↙ ↘ Both ends down

Example 1: Determine the end behavior of the function:

a) $f(x) = x^4 - 5x^2 + 4.$

• degree = 4 even
positive coefficient $\Rightarrow$ ↖ ... ↗
$(x^2$ like$)$

b) $f(x) = -4x - x^2 + 2x^3 - x^5.$

• degree = 5 odd
negative coefficient $\Rightarrow$ ↖ ... ↘
$(-x^3$ like$)$

c) $f(x) = 2x^{1}(x-1)^2(2x-1)^3(4-x)^{1}.$

degree = 1 + 2 + 3 + 1 = 7 odd
leading coefficient $= 2 \cdot 1^2 \cdot 2^3 \cdot (-1)$
$= -16$ negative $\Rightarrow$ ↖ ... ↘
$(-x^3$ like$)$

Next, you should be able to find the *x* intercept(s) and the *y* intercept of a polynomial function.

You will need to set the function equal to zero and then use **the Zero Product Property** to find the *x* intercept(s). That means if $ab = 0$, then either $a = 0$ or $b = 0$. To find the *y* intercept of a function, you will find $f(0)$.

(look for graph on next slide)

Example 2: Find the x and y intercept(s) of $f(x) = -2(x-3)(x+4)(2-x)$.

- x-intercepts : $f(x)=0 \Rightarrow x-3=0$, $x+4=0$, $2-x=0$ (repeated like a line once)
  $\boxed{x = 3, -4, 2} \Leftrightarrow$
- y-intercept $= f(0) = -2(0-3)(0+4)(2-0) = 48 \Rightarrow \boxed{(0,48)}$

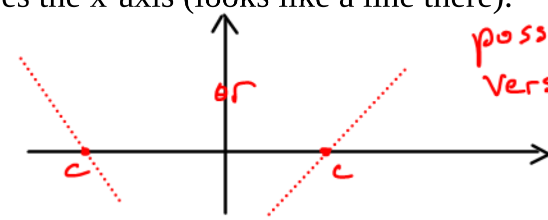
degree $= 1+1+1 = 3$, leading coefficient $= -2 \cdot 1 \cdot 1 \cdot (-1) = 2$ positive.

In some problems, one or more of the factors will appear more than once when the function is factored. The power of a factor is called its multiplicity. So given $P(x) = x^2(x-3)^3(x+1)$, then the multiplicity of the first factor is 2, the multiplicity of the second factor is 3 and the multiplicity of the third factor is 1.

If the **multiplicity of a factor is 1:** the graph crosses the x-axis (looks like a line there).

the factor has degree 1

$(x-c)^1 \to (x-c)$

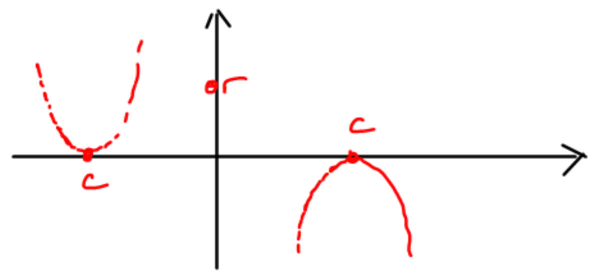


possible versions

If the **multiplicity of a factor is even**: the graph touches the x-axis, but does not cross it (looks like a parabola there).

the factor has even degree:

$(x-c)^2$ or $(x-c)^4$ or ...



If the **multiplicity of a factor is odd and greater than 1:** the graph crosses the x-axis and it looks like a cubic there.

the factor has odd degree:

$(x-c)^3$ or $(x-c)^5$ or ...

Example 3: Find the x intercept(s) of the function and state the multiplicity of each. Indicate the possible behavior of the graph through each zero: (graph on next slide)

$f(x) = -2(x-1)^2(x+4)(x+5)^3$

- x-intercepts : $f(x)=0 \Rightarrow x-1=0$, $x+4=0$, $x+5=0$
  $x=1$ (parabola) $x=-4$ (line) $x=-5$ (cubic)
- y-intercept $= f(0) = -2(0-1)^2(0+4)(0+5)^3 = -1000 \Rightarrow \boxed{(0,-1000)}$

degree $= 2+1+3 = 6$ even, coeff $= -2 \cdot 1^2 \cdot 1 \cdot 1^3 = -2$ negative $\swarrow \dots \searrow$

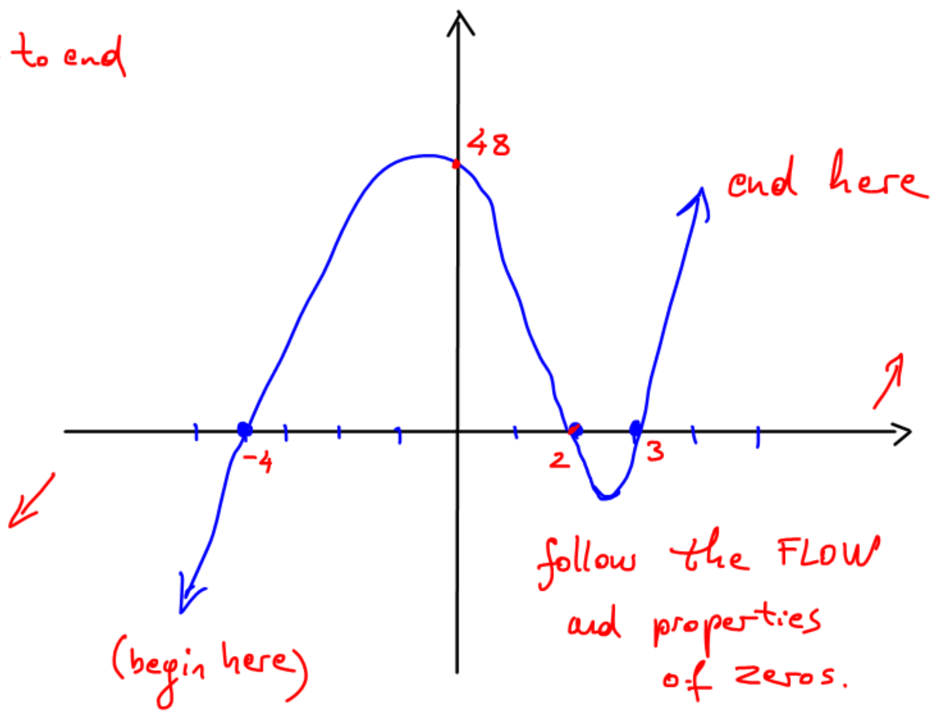
example 1. graph for $f(x) = -2(x-3)(x+4)(2-x)$.

$\left\{ \begin{array}{l} \text{deg} = 3 \\ \text{coeff.} = \oplus \end{array} \right.$ $\quad \swarrow \cdots \nearrow$ ← how to end

← how to begin

$\left. \begin{array}{l} x = 3 \\ x = -4 \\ x = 2 \end{array} \right\}$ line behaviour (plot the zeros)

$\{\ y = f(0) = 48$



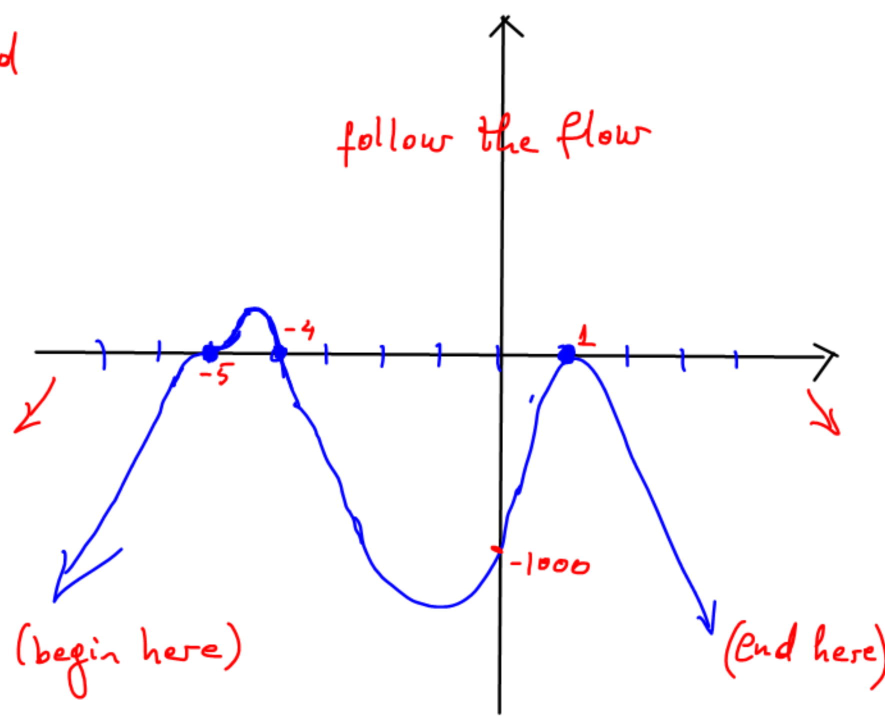
example 3. $f(x) = -2(x-1)^2(x+4)(x+5)^3$

$\left\{ \begin{array}{l} \text{degree} = 6 \\ \text{coeff.} = \ominus \end{array} \right.$ $\quad \swarrow \cdots \searrow$ ← how to end

↑ how to begin

$\left\{ \begin{array}{l} x = 1 \text{ (parabola-tangent)} \\ x = -4 \text{ (line)} \\ x = -5 \text{ (cube)} \end{array} \right.$ plot the zeros

$\{\ y = f(0) = -1000$



Now we'll put all of this information together to generate the graph of a polynomial function. For each problem, you'll need to state

Follow these steps before graph

- the degree of the function
- the leading coefficient of the function
- the end behavior of the function
- the x and y intercepts (and multiplicities)
- behavior of the function through each of the x intercepts (zeros) of the function

Your graph should be smooth, with no sharp corners.
Note that graphs of polynomial functions may have peaks or valleys, but without additional information, you will not be able to determine how high or low these points are.

Example 4: Find the x and y intercepts of the graph of the function. State the multiplicities of the zeros of the function. State the degree of the function and find the leading coefficient. Indicate the end behavior of the function and the behavior of the function through each zero. Use all of this information to graph the function.

$P(x) = (x+2)^1(x-1)^3(x-4)^2$.

- behaviour

degree $= 1+3+2 = 6$ even

coeff. $= 1 \cdot 1^3 \cdot 1^2 = 1$ ⊕

$\Rightarrow \nwarrow \ldots \nearrow$

- x-intercepts

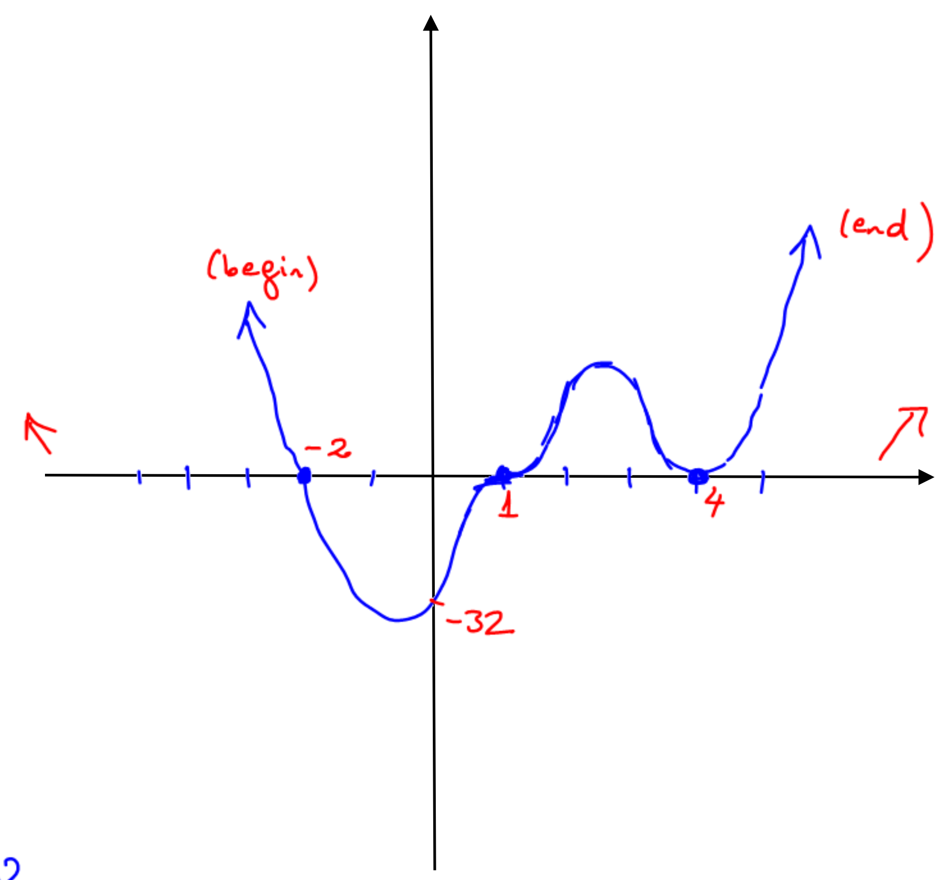
$x+2=0 \Rightarrow x=-2$ (line)

$x-1=0 \Rightarrow x=1$ (cube)

$x-4=0 \Rightarrow x=4$ (parabola)

- y-intercept

$y = f(0) = (0+2)(0-1)^3(0-4)^2 = -32$



Example 5: Write the equation of the cubic polynomial $P(x)$ with leading coefficient -2 whose graph passes through (2, 16) and is tangent to the x-axis at the origin.

→ cubic function → should have 3 factors

→ $a = -2$

→ $(2, 16)$ – given point

→ tangent to x-axis at origin
$x = 0$, mult. 2 – parabola shapes

$$f(x) = -2(x-0)^2\underbrace{(x-c)}_{\text{the other factor}}$$

$$f(x) = -2x^2(x-c)$$

$$f(2) = -2\cdot 2^2(2-c) = 16 \Rightarrow c = 4$$

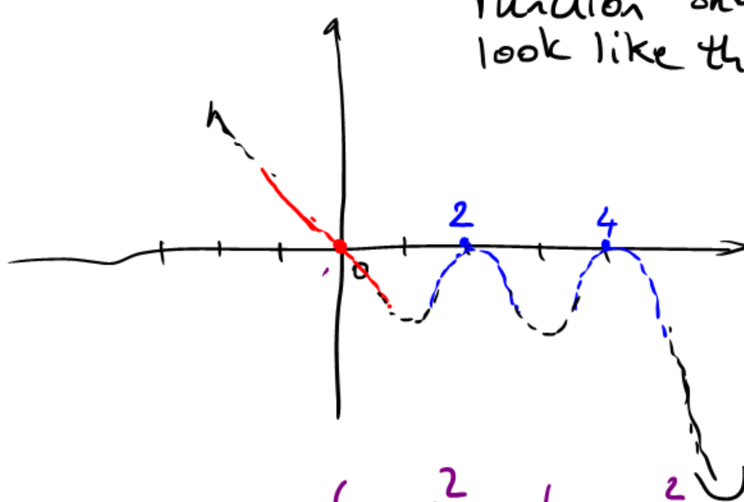
$$\Rightarrow \boxed{f(x) = -2x^2(x-4)}$$

exercise

Example 6: Write the equation of a 5^{th} degree polynomial with leading coefficient -1 given that the graph of the polynomial is tangent to the x-axis at the points 2 and 4 and the graph passes through the origin.

Being 5^{th} degree = odd
and ⊖ leading coefficient,
end behaviour ↖ ... ↘

Function should look like this



$x = 0$ mult. 1

$x = 2$ mult. 2

$x = 4$ mult. 2

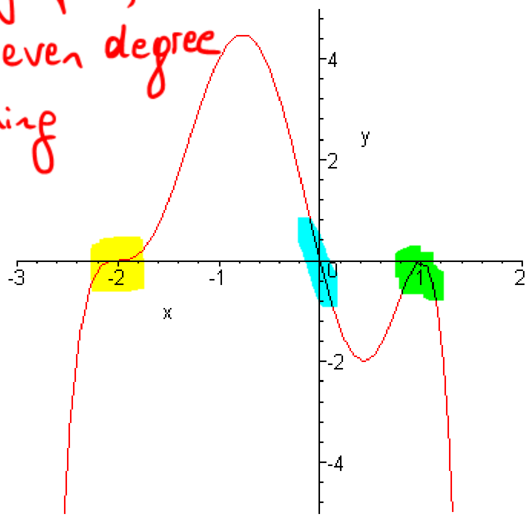
$$x \cdot (x-2)^2 \cdot (x-4)^2$$

$$\Longrightarrow f(x) = -x(x-2)^2(x-4)^2$$

Example 7: Given the graph of a polynomial, try to determine the equation of the polynomial.

By graph, it is even degree

$\ominus$ leading



Look at the zeros, i.e. x-intercepts:

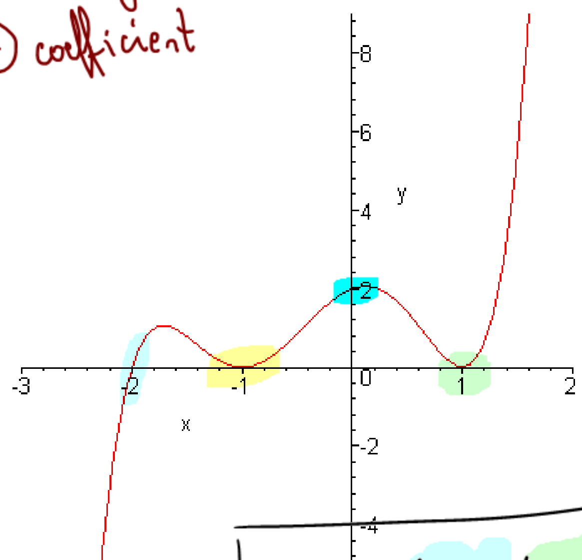
- $x=-2 \Leftrightarrow$ cubic $\Leftrightarrow (x+2)^3$
- $x=0 \Leftrightarrow$ line $\Leftrightarrow x$
- $x=1 \Leftrightarrow$ parabola $\Leftrightarrow (x-1)^2$

$$\boxed{f(x) = -x\cdot(x+2)^3\cdot(x-1)^2}$$

By behaviour ↙…↗

{ odd degree

{ $\oplus$ coefficient



- x-int:

$x=-2 \Leftrightarrow$ line $\Leftrightarrow (x+2)$

$x=-1 \Leftrightarrow$ parabola $\Leftrightarrow (x+1)^2$

$x=1 \Leftrightarrow$ parabola $\Leftrightarrow (x-1)^2$

$\Rightarrow f(x) = a(x+2)(x+1)^2(x-1)^2$

- y-int, $f(0)=2$

$f(0) = a(0+2)(0+1)^2(0-1)^2 = 2$

$a = 1$

$$\Rightarrow \boxed{f(x) = (x+2)(x+1)^2(x-1)^2}$$

We may not have enough time to solve all the examples here. Since graphing polynomials is a subject covered in College Algebra, we assume that you are already familiar with this subject. If you are not comfortable with graphing polynomials, please study! You can check Chapter 4 of the online textbook for College Algebra (the link is on your CASA account!). Please solve these extra problems to practice.

(Extra) Example: Find the x and y intercepts of the graph of the function. State the multiplicities of the zeros of the function. State the degree of the function and find the leading coefficient. Indicate the end behavior of the function and the behavior of the function through each zero. Use all of this information to graph the function.

$P(x) = (x+3)^2(x-1)^2(2-x)^1.$

degree $= 2+2+1 = 5$ odd

coefficient $= \ominus$

behaviour $\nwarrow \ldots \searrow$

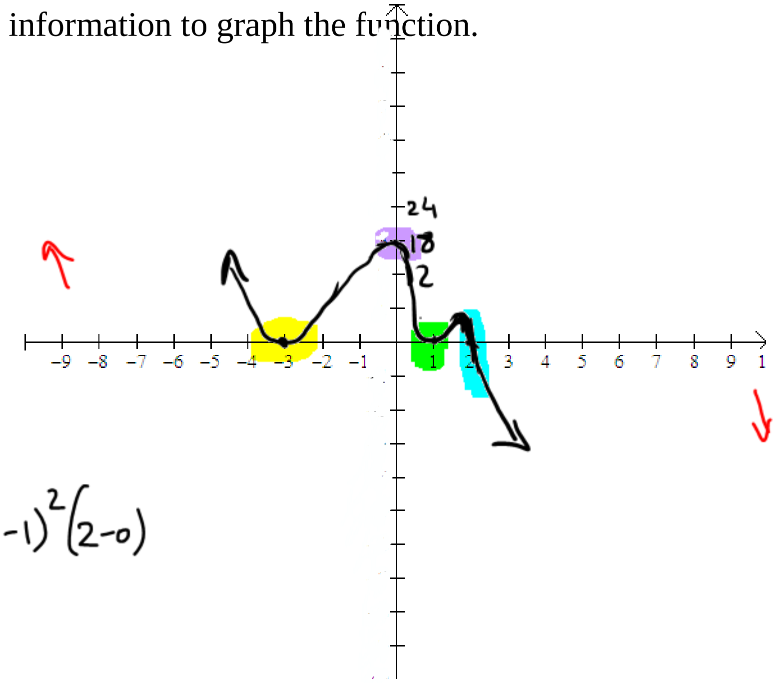
$x = -3$ (parabola)

$x = 1$ (parabola)

$x = 2$ (line)

$P(0) = (0+3)^2(0-1)^2(2-0)$

$P(0) = 18$



(Extra) Example: Find the x and y intercepts of the graph of the function. State the multiplicities of the zeros of the function. State the degree of the function and find the leading coefficient. Indicate the end behavior of the function and the behavior of the function through each zero. Use all of this information to graph the function.

$P(x) = x^3 - 4x^2 - x + 4.$ (Factor first!!!)

$P(x) = x^2(x-4) - (x-4)$

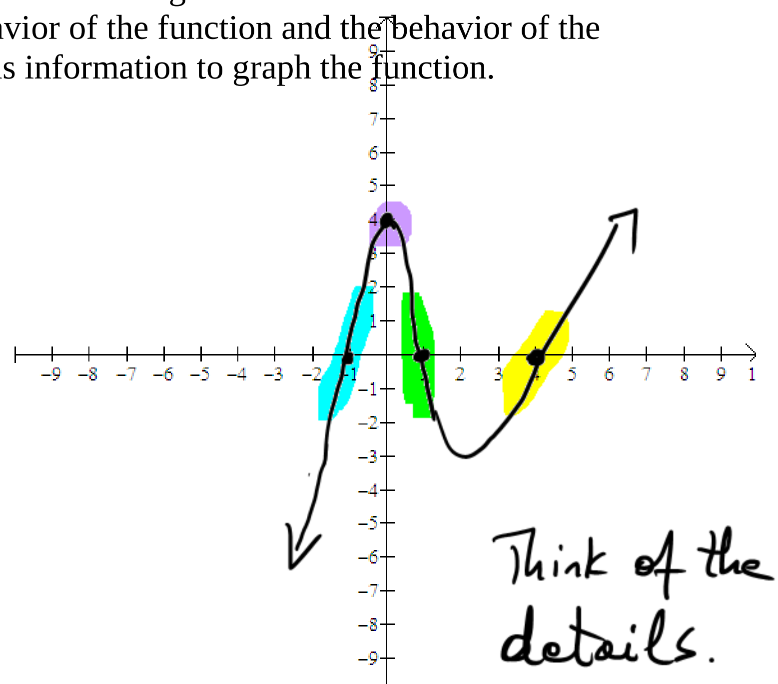
$= (x-4)(x^2-1)$

$P(x) = (x-4)(x+1)(x-1)$

$\rightarrow$ 3rd degree, $\oplus$

$\swarrow \ldots \nearrow$

y-int: $P(0) = 4$



Think of the details.

(Extra) Example: Find the x and y intercepts of the graph of the function. State the multiplicities of the zeros of the function. State the degree of the function and find the leading coefficient. Indicate the end behavior of the function and the behavior of the function through each zero. Use all of this information to graph the function.

$P(x) = (4-x)(x-1)^2(x+5)^3$.

deg $= 1+2+3 = 6$ even

leading $= -1$

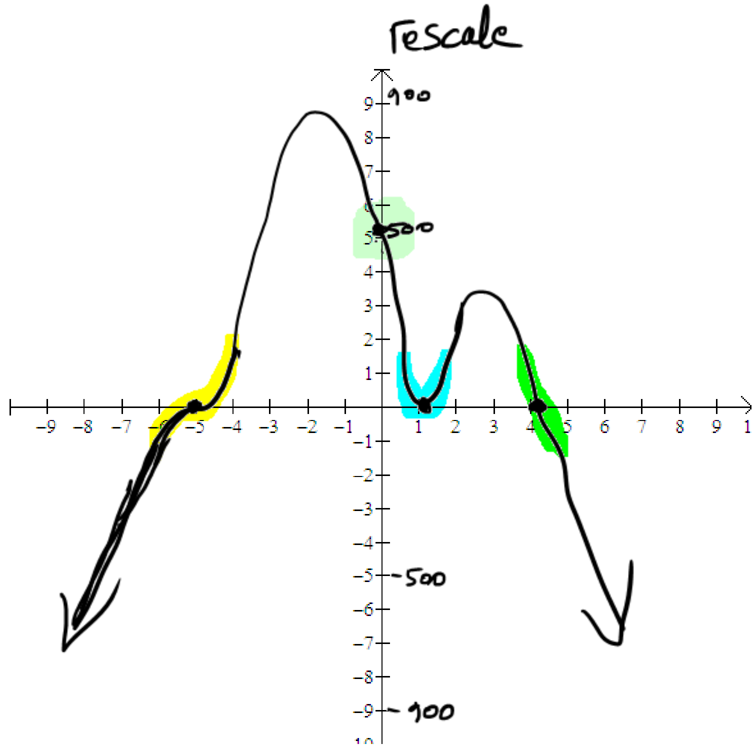
↙ ⋯ ↘

x-int: 4, 1, −5

line parabola cube

y-int: $P(0) = (4-0)(0-1)^2(0+5)^3$

$P(0) = 500$



NOTE: With some problems, you can use transformations to graph polynomial functions.

(Extra) Example: Graph using transformations: $f(x) = (x-1)^3 - 4$

① $y = x^3$

② ↳ shift 1 right

③ ↳ shift 4 down