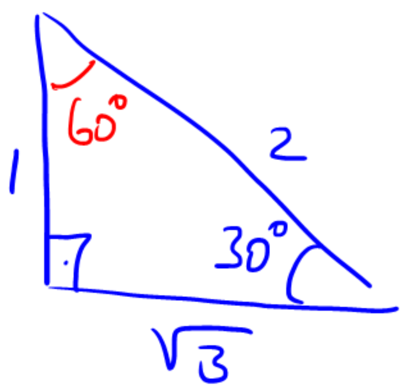
We learned six trigonometric functions that are defined for **angles** in a right triangle.



$$\sin(30^\circ) = \frac{1}{2}$$

$$\sin(60^\circ) = \frac{\sqrt{3}}{2}$$

As the value of angles change, the trigonometric function will change. Thus, we need to explore **ANGLES**.

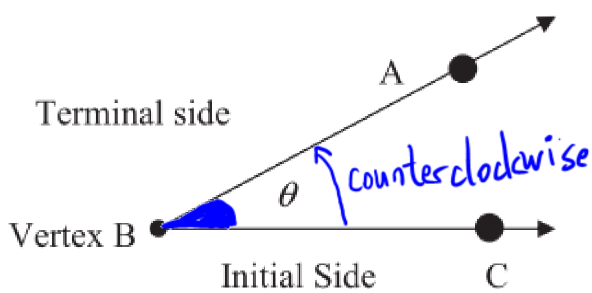
To be continued on Tuesday, 03/01.

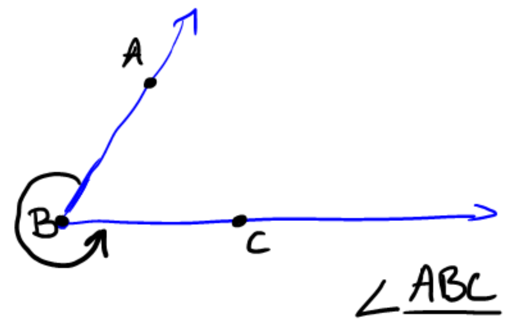
Math 1330 - Section 4.2
Radians, Arc Length, and Area of a Sector

The word trigonometry comes from two Greek roots, trigonon, meaning "having three sides," and meter, meaning "measure." We have already defined the six basic trigonometric functions in terms of a right triangle and the measures of its three sides.

Before beginning our study of trigonometry, we need to take a look at some basic concepts having to do with angles. ← study

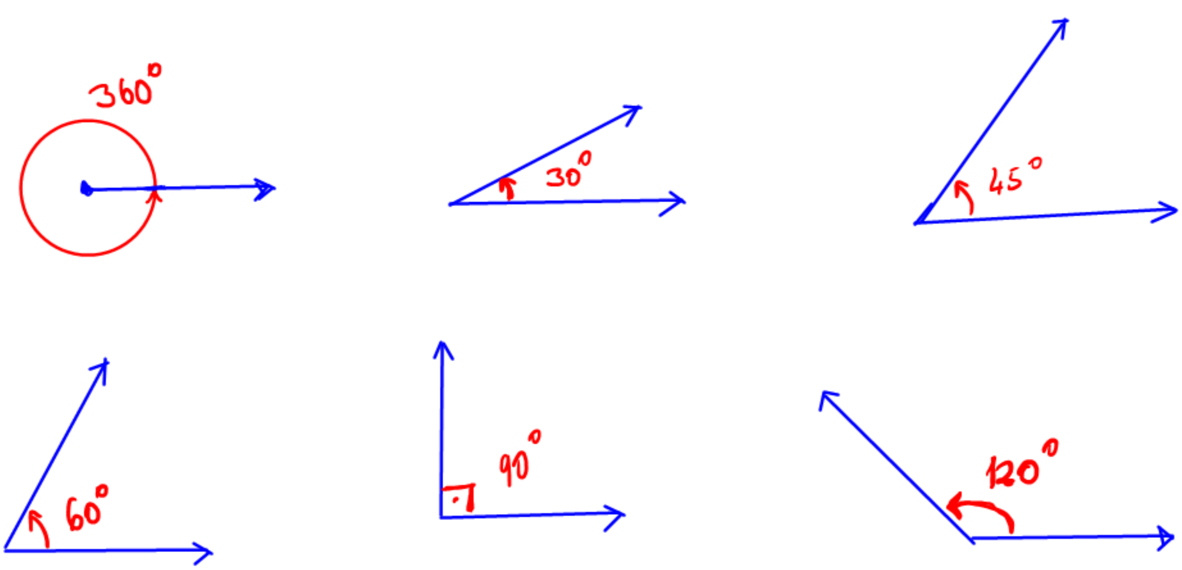
An angle is formed by two rays that share a common endpoint, called the **vertex** of the angle. One ray is called initial side of the angle, and the other side is called the terminal side. For ease, we typically will draw angles in the coordinate plane with the initial side along the positive *x* axis.





We measure angles in two different ways, both of which rely on the idea of a complete revolution in a circle.

You are probably familiar with degree measure. In this system of angle measure, an angle which is one complete revolution is 360°. So one degree is $\frac{1}{360}^{\text{th}}$ of a circle.

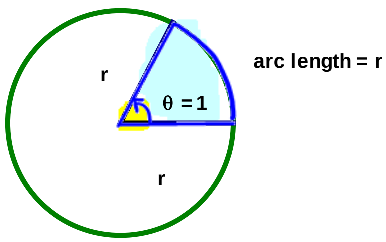


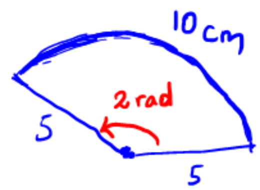
The second method is called radian measure. One complete revolution is 2π.

$360^\circ = 2\pi$ one revolution

Suppose I draw a circle and construct an angle by drawing rays from the center of the circle to two different points on the circle in such a way that the length of the arc intercepted by the two rays is the same as the radius of the circle. The measure of the central angle thus formed is **one radian.**

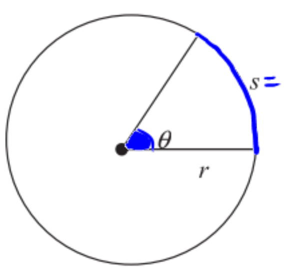
$\approx 60^\circ$ (almost $\frac{1}{6}$ of whole circle)





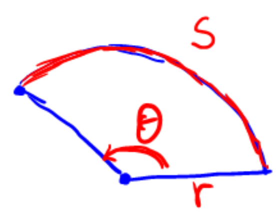
Radian measure of an angle:

Place the vertex of the angle at the center of a circle of radius *r*. Let *s* denote the length of the arc intercepted by the angle. The **radian measure** θ of the angle is the ratio of the arc length *s* to the radius *r*. That is, $\theta = \frac{s}{r}$.



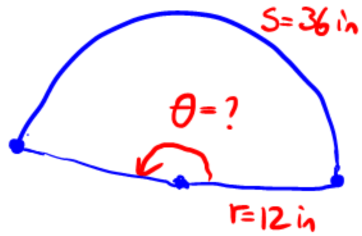
arclength btwn two sides of angle

$$\boxed{\theta = \frac{s}{r}}$$



In general, the **radian measure of a central angle** θ can be determined by the formula $\boxed{\theta = \frac{s}{r}}$, where *s* is the length of the intercepted arc and *r* is the radius of the circle and *r* and *s* are measured in the same units.

Example 1: A circle has radius 12 inches. A central angle θ intercepts an arc of length 36 inches. What is the radian measure of θ?



$\theta = ?$

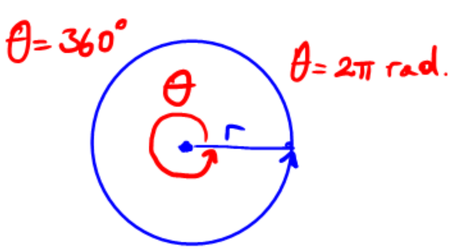
$\Rightarrow \theta = \frac{s}{r} = \frac{36}{12} = \underline{3 \text{ rad}}$.

$\boxed{\text{Answer: } \theta = 3 \text{ rad}}$

full revolution

We know that the circumference of a circle is $2\pi r$. In this case, $\boxed{\theta = \frac{2\pi r}{r} = 2\pi}$. So the radian measure of the central angle in the case of a complete revolution is 2π.

Comparing the two systems, then, we have that



2π radians = 360   for short $2\pi = 360°$

π radians = 180 °   or $\pi = 180°$

$\frac{\pi}{2}$ radians = 90 °   or $\frac{\pi}{2} = 90°$

etc.

As you are becoming more familiar with radian measure, you may find it helpful to be able to convert between the two systems. We can use the statement π radians = 180 ° to help do this. Dividing both sides of that equation by π, we have that

1 radian = $\frac{180°}{\pi}$ so **to convert to degrees, multiply by** $\frac{180°}{\pi}$.

Similarly,

1 degree = $\frac{\pi}{180°}$ so **to convert to radians, multiply** by $\frac{\pi}{180°}$.

These are the conversion formulas for radians to degrees and for degrees to radians, respectively.

If π rad = 180° $\Rightarrow$ $\boxed{1 \text{ rad} = \frac{180°}{\pi}}$.

If 180° = π rad $\Rightarrow$ $\boxed{1° = \frac{\pi}{180} \text{ rad.}}$

} Very Important.

1 radian = $\frac{180°}{\pi}$ so **to convert to degrees, multiply by** $\frac{180°}{\pi}$.

1 degree = $\frac{\pi}{180°}$ so **to convert to radians, multiply** by $\frac{\pi}{180°}$.

Let's practice!!!

Example 2: Convert 135° to radian measure.

degrees to radians : $135° = \cancel{135}^{3} \times \frac{\pi}{\cancel{180}_{4}} = \frac{3\pi}{4}$ rad.

Note:

Example 3: Convert $\frac{4\pi}{3}$ to degrees.

$\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

$= 180° + 60° = 240°$

radians to degree : $\frac{4\pi}{3} = \frac{4\cancel{\pi}}{\cancel{3}} \times \frac{\cancel{180}^{60}}{\cancel{\pi}} = 240°$.

Example 4: Convert $\frac{2\pi}{9}$ to degrees.

radians to degree : $\frac{2\pi}{9} = \frac{2\cancel{\pi}}{9} \times \frac{180}{\cancel{\pi}} = 40°$.

Example 5: Convert 18° to radian measure.

degrees to radians : $18° = \cancel{18} \times \frac{\pi}{\cancel{180}_{10}} = \frac{\pi}{10}$ rad.

You will use some angles so often that you should know both their degree and radian measures. These are:

$30° = \frac{\pi}{6}$ $\longleftrightarrow$ $\frac{\pi}{6} = 30°$

$45° = \frac{\pi}{4}$ $\longleftrightarrow$ $\frac{\pi}{4} = 45°$

$60° = \frac{\pi}{3}$ $\longleftrightarrow$ $\frac{\pi}{3} = 60°$

$90° = \frac{\pi}{2}$ $\longleftrightarrow$ $\frac{\pi}{2} = 90°$

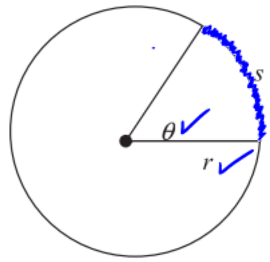
$180° = \pi$ $\longleftrightarrow$ $\pi = 180°$

$360° = 2\pi$ $\longleftrightarrow$ $2\pi = 360°$

You should know these angles in radian measure.

Memorize these!

Angles will mostly be used in radian measure.



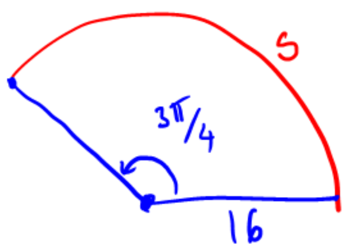
Question: If the angle is given and the radius is known, can you find the arclength? Yes.

If $\boxed{\theta = \frac{s}{r}}$, then we can multiply both sides of this equation by r, which gives us $\boxed{s = r\theta}$.

This is called the **arclength formula** and it gives the length of the arc intercepted by the central angle. Note, to use this formula the angle measure MUST be given in radians.

Never forget!!!

Example 6: If the radius of a circle is 16 inches and the measure of its central angle is $\frac{3\pi}{4}$, find the arclength of the sector intercepted by the angle.

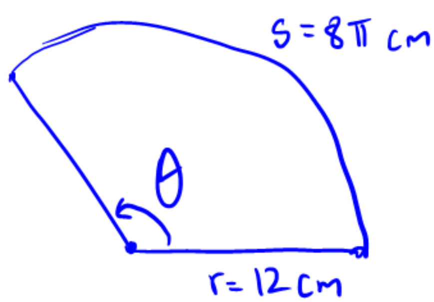


$\Rightarrow$ $s = \theta \cdot r$, θ should be in radian. ✓

$= \frac{3\pi}{4} \times 16$

$\boxed{s = 12\pi \text{ in.}}$

Example 7: If the arclength of a sector is 8π cm. and the radius is 12 cm., find the measure of the central angle.



$\Rightarrow$ $\theta = \frac{s}{r} = \frac{8\pi}{12} = \frac{2\pi}{3}$

$\boxed{\theta = \frac{2\pi}{3} \text{ rad}}$

ex. Find the arclength of the sector intercepted by the central angle $\theta = 60°$, and radius $r = 3$ cm.

Solution:

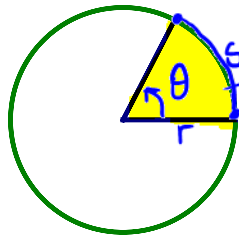
$$\theta = \frac{s}{r} \iff \boxed{s = r \cdot \theta}$$

θ should be in radians.

$$\theta = 60° = \cancel{60} \times \frac{\pi}{\cancel{180}_{3}} = \frac{\pi}{3} \text{ rad.}$$

$$\Rightarrow \boxed{s = r \cdot \theta = 3 \cdot \frac{\pi}{3} = \pi \text{ cm}}$$

A **sector of a circle** is the region bounded by a central angle and the intercepted arc.



Find the area of this sector!

Sometimes, you'll need to find the **area of a sector**. The formula for the area of a circle is $A = \pi r^2$. A sector is a fraction of a circle, determined by the measure of its central angle over the complete revolution that is a circle, that is $\frac{\theta}{2\pi}$. So the area of a section is this fraction of the area of the circle, that is:

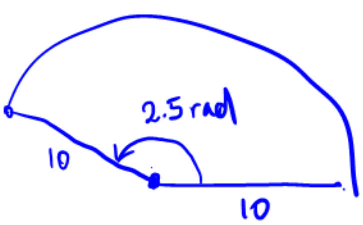
$$\boxed{A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{r^2\theta}{2} = \frac{1}{2} r^2 \theta.}$$

$\Rightarrow \quad A = \frac{1}{2} r^2 \cdot \theta$

Note, to use this formula, the measure of the central angle must be given in radians.

Do not forget "RADIAN".

Example 8: A sector has radius 10 and central angle measuring 2.5 radians. Find the area of the sector.



$\Rightarrow \quad A = \frac{1}{2} r^2 \cdot \theta = \frac{1}{2} \cdot 10^2 \cdot 2.5 = 125$

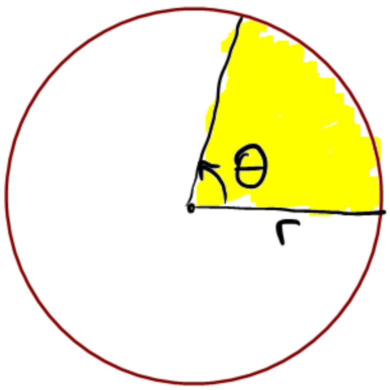
$$\boxed{A = 125 \text{ unit}^2}$$

* * * If the angle is in degrees, then area of sector can be found by

$$A = \pi r^2 \cdot \frac{\theta^\circ}{360^\circ}.$$

degrees

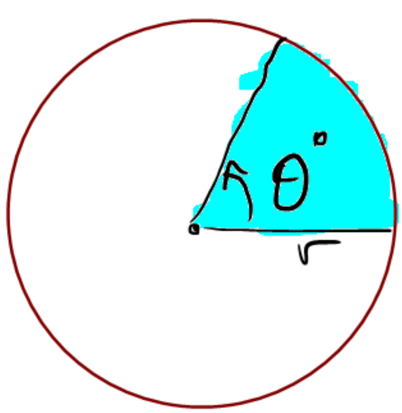
- Area of sector with angle in radian measure.



- Area of circle $= \pi \cdot r^2$
- Sector covers $\frac{\theta}{2\pi}$ part of circle

$$\Rightarrow \text{Area}_{\text{sector}} = \frac{\theta}{2\cancel{\pi}} * \cancel{\pi} r^2 = \boxed{\frac{1}{2}\theta \cdot r^2}$$

- Area of sector with angle in degree measure



- Area of circle $= \pi r^2$
- Sector is $\frac{\theta}{360°}$ part of circle

$$\Rightarrow \text{Area}_{\text{sector}} = \boxed{\frac{\theta}{360} * \pi r^2}$$

Example 9: A sector has central angle measuring 5 radians. The area of the sector is 2500 square units. Find the radius.

We know:

$A = 2500 \text{ unit}^2$

$\theta = 5 \text{ rad}$

$\Rightarrow r = ?$

$\Rightarrow A = \frac{1}{2} \cdot r^2 \cdot \theta$

$2500 = \frac{1}{2} \cdot r^2 \cdot 5 \Rightarrow 5000 = r^2 \cdot 5$

$r^2 = 1000 \rightarrow \boxed{r = 10\sqrt{10} \text{ unit}}$

Example 10: Find $\underbrace{\text{the perimeter of a sector}}$ with central angle $60°$ and radius 3 m.

We know:

$\theta = 60° = 60 \cdot \frac{\pi}{180} = \frac{\pi}{3} \text{ rad}$

$r = 3\text{m}$

Find Perimeter $= r + r + s$

- r ✓
- $s = r \cdot \theta = 3 \cdot \frac{\pi}{3} = \pi$ ✓

$\boxed{\text{Perimeter} = 6 + \pi}$

Example 11: If the area of a sector is 2π m and the measure of the central angle is $\frac{\pi}{4}$, find the radius.

We know:

$A = 2\pi$

$\theta = \frac{\pi}{4}$

$\Rightarrow r = ?$

$\Rightarrow A = \frac{1}{2} r^2 \cdot \theta$

$2\pi = \frac{1}{2} r^2 \cdot \frac{\pi}{4}$

$16\cancel{\pi} = \cancel{\pi} \cdot r^2$

$r^2 = 16 \Rightarrow \boxed{r = 4\text{m}}$

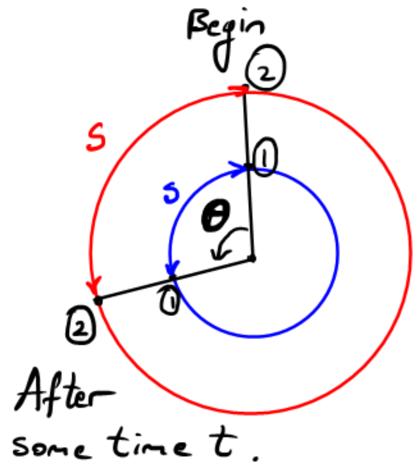
Popper $\underline{\underline{13}}$: AAA

Angular and Linear Velocity

Suppose you are riding on a merry-go-round.

Angular Velocity $\omega = \dfrac{\theta}{t}$

Linear Velocity $v = \dfrac{s}{t}$



• student 1 and 2 cover same angle,
• but their arclength is different

The ride travels in a circular motion, and the horses usually move up and down.

Some of the horses are right along the edge of the merry-go-round, and some are closer to the center. If you are on one of the horses at the edge, you will travel farther than someone who is on a horse near the center. But the length of time that both people will be on the ride is the same. If you were on the edge, not only did you travel farther, you also traveled faster.

However, everyone on the merry-go-round travels through the same number of degrees (or radians).
There are two quantities we can measure from this, angular velocity and linear velocity.

The ***angular velocity*** of a point on a rotating object is the number of degrees (or radians or revolutions) per unit of time through with the point turns. This will be the same for all points on the rotating object.

The ***linear velocity*** of a point on the rotating object is the distance per unit of time that the point travels along its circular path. This distance will depend on how far the point is from the axis of rotation (the center of the merry-go-round).

We let the Greek letter ω represent angular velocity. Using the definition above,

$$\omega = \frac{\theta}{t}$$

We denote linear velocity by v. Using the definition above, $v = \dfrac{a}{t}$, where a is the arclength.

The relationship between these two quantities is given by $v = r\omega$, where r is the radius.

$$v = \frac{s}{t} = \frac{r \cdot \theta}{t} = r \cdot \frac{\theta}{t} = r \cdot \omega$$

angular → $\omega = \dfrac{\theta}{t}$ linear → $v = \dfrac{a}{t}$ $v = r\omega$

Example 12: If the speed of a revolving gear is 25 rpm,

1 rotation = 360°

a. Find the $\underbrace{\text{number of degrees per minute}}$ through which the gear turns.

angular velocity = 25 rpm

$= 25 \times 360^\circ = \boxed{9000^\circ/\text{min} = \omega}$

b. Find the $\underbrace{\text{number of radians per minute}}$ through which the gear turns.

1 rotation = 2π rad

angular velocity = 25 rpm

$= 25 \times 2\pi = \boxed{50\pi \text{ rad}/\text{min} = \omega}$

Example 13: A car has wheels with a 10 inch radius. If each wheel's rate of turn is 4 revolutions per second,

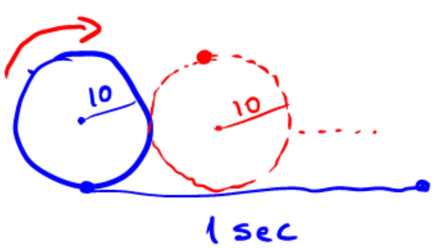
1 revolution = 2π

a. Find the angular speed in units of radians/second.

$\omega = 4 \text{ rps} = 4 \cdot 2\pi = \boxed{8\pi \text{ rad}/\text{sec}}$

b. How fast (linear speed) is the car moving in units of inches/second?

$V = r \cdot \omega$



$V = r \cdot \omega = 10 \times 8\pi$

$\boxed{V = 80\pi \text{ in}/\text{sec}}$

Example 14: A CD spins at the rate of 500 revolutions per minute. How many degrees per minute is this?

exercise

1 revolution $= 360^\circ$

$\omega = 500 \text{ rpm}$

$\omega = 500 \times 360^\circ$

$\omega = 180{,}000^\circ/\text{min}$