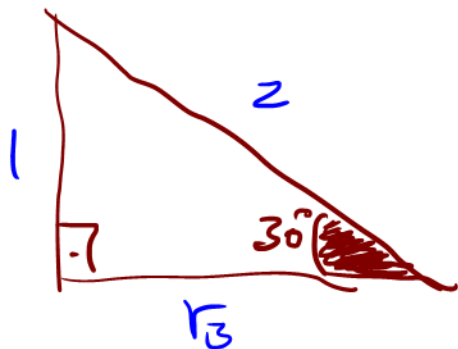


Popper # 09

Given the right triangle,



$$\sin 30^\circ = \frac{1}{2} \quad \csc 30^\circ = 2$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} \quad \sec 30^\circ = \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \quad \cot 30^\circ = \sqrt{3}$$

① $\sin 30^\circ$ D

A. $\sqrt{3}$

② $\cos 30^\circ$ B

B. $\sqrt{3}/2$

③ $\tan 30^\circ$ C

C. $\frac{\sqrt{3}}{3}$

④ $\cot 30^\circ$ A

D. $\frac{1}{2}$

⑤ $\sec 30^\circ$ E

E. 2

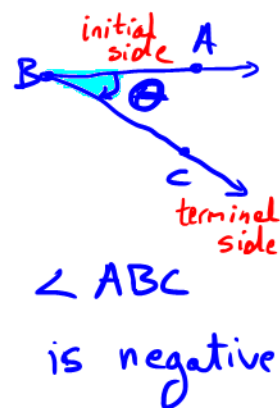
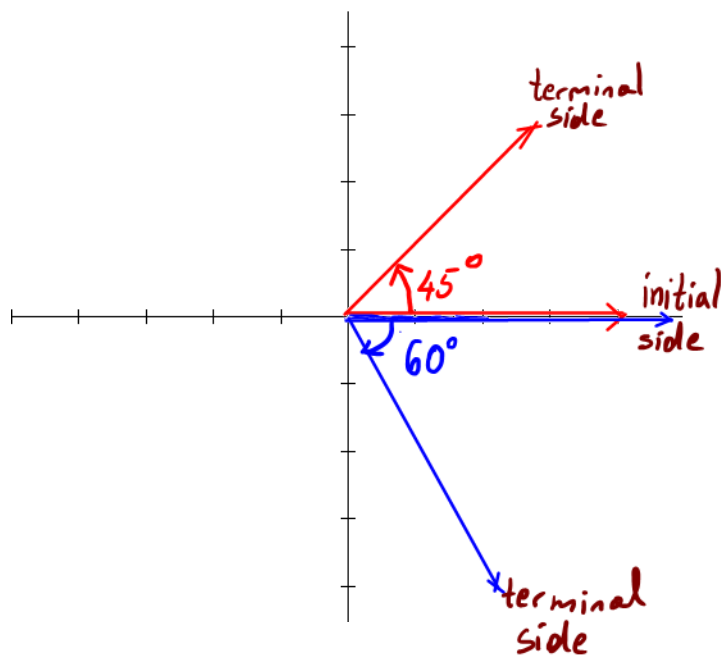
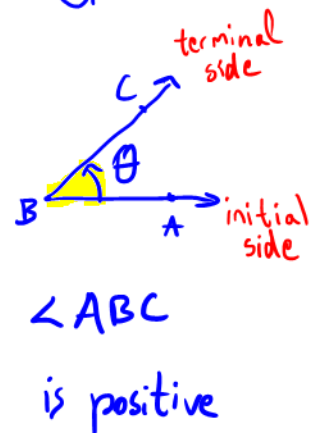
⑥ $\csc 30^\circ$ E

E. $\frac{2\sqrt{3}}{3}$

} call both E.

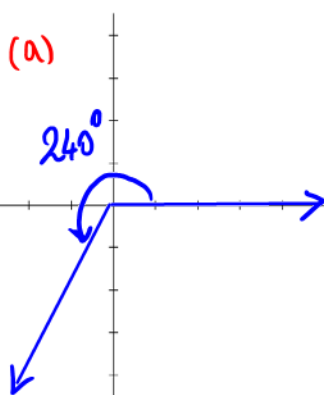
Math 1330 - Section 4.3 Unit Circle Trigonometry

An angle is **in standard position** if its vertex is at the origin and its initial side is along the positive x axis. **Positive angles** are measured **counterclockwise** from the initial side. **Negative angles** are measured **clockwise**. We will typically use the Greek letter θ to denote an angle.

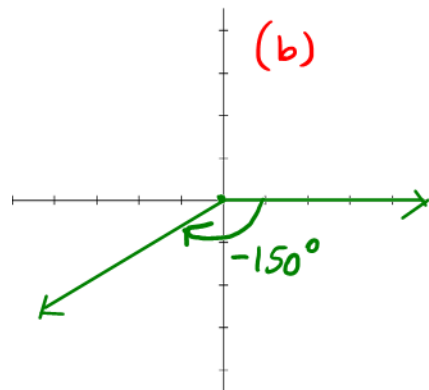


Example 1: Sketch each angle in standard position.

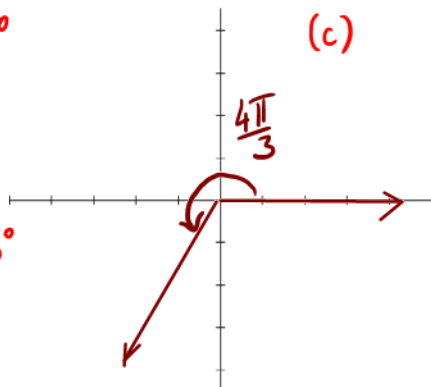
a. $240^\circ = 180^\circ + 60^\circ$
 $240^\circ = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$



b. $-150^\circ = -90^\circ - 60^\circ$
 $-150^\circ = -\frac{\pi}{2} - \frac{\pi}{3} = -\frac{5\pi}{6}$

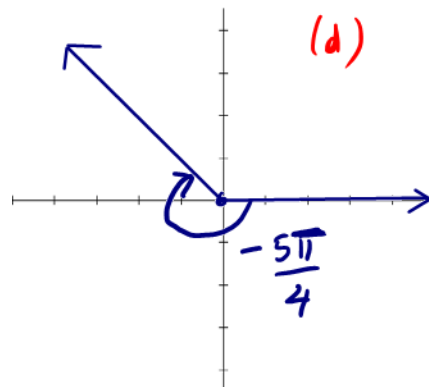


c. $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$
 $\frac{4\pi}{3} = 180^\circ + 60^\circ = 240^\circ$



d. $-\frac{5\pi}{4} = -\pi - \frac{\pi}{4}$

$-\frac{5\pi}{4} = -180^\circ - 45^\circ = -225^\circ$

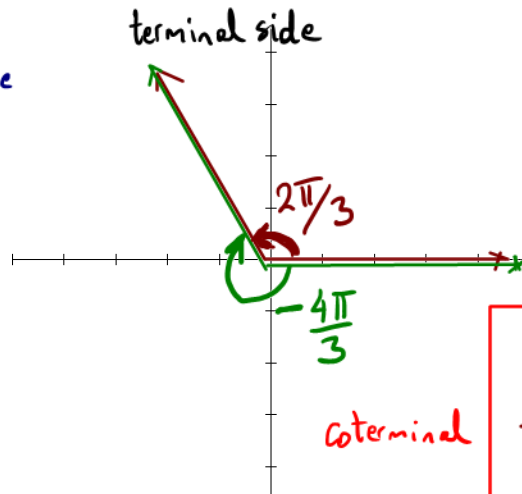
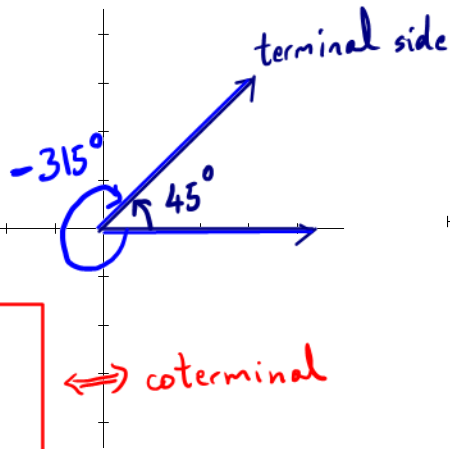


Angles that have the same terminal side are called **coterminal angles**. Measures of coterminal angles differ by a multiple of 360° if measured in degrees or by a multiple of 2π if measured in radians.

Demonstration

Both 45°
and -315°
have same
terminal side.

$$\begin{aligned} 45^\circ &= -315^\circ + 360^\circ \\ -315^\circ &= 45^\circ - 360^\circ \end{aligned}$$



Both $\frac{2\pi}{3}$ & $-\frac{4\pi}{3}$
have same
terminal side.

$$\begin{aligned} \frac{2\pi}{3} &= -\frac{4\pi}{3} + 2\pi \\ -\frac{4\pi}{3} &= \frac{2\pi}{3} - 2\pi \end{aligned}$$

Example 2: Find three angles, two positive and one negative that are coterminal with each angle.

a. $512^\circ = 360^\circ + 152^\circ$
 $512^\circ = \boxed{152^\circ} + \underbrace{360^\circ}$

$$\begin{aligned} \theta_1 &= 152^\circ = 512^\circ - 360^\circ \\ \theta_2 &= 512^\circ + 360^\circ = 872^\circ \\ \theta_3 &= 152^\circ - 360^\circ = -208^\circ \end{aligned} \quad \left. \vphantom{\begin{aligned} \theta_1 \\ \theta_2 \\ \theta_3 \end{aligned}} \right\} \text{Coterminal}$$

b. $\frac{-15\pi}{8}$

$$\theta_1 = -\frac{15\pi}{8} + 2\pi = -\frac{15\pi}{8} + \frac{16\pi}{8} = \frac{\pi}{8} (+)$$

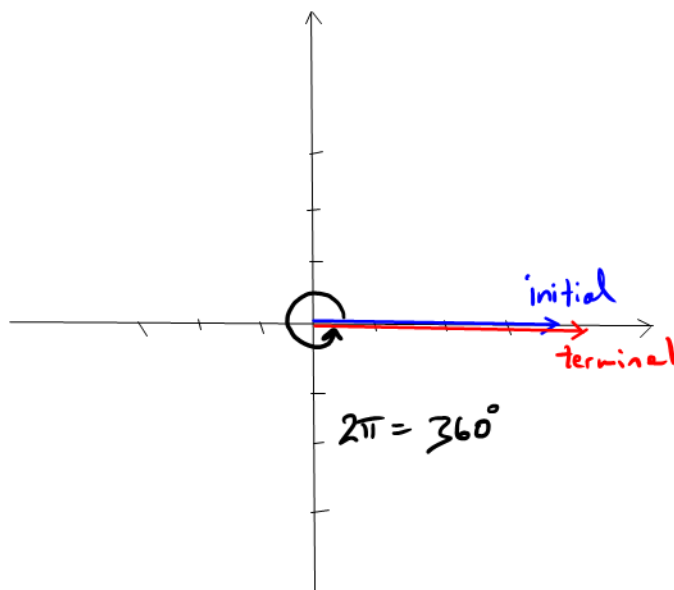
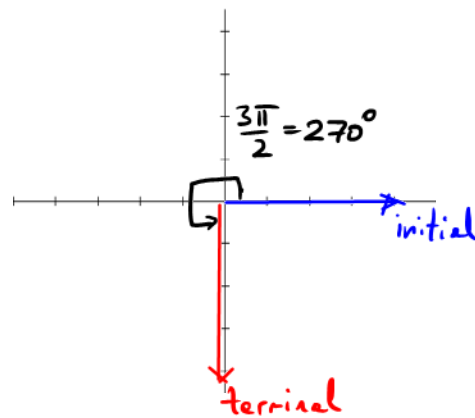
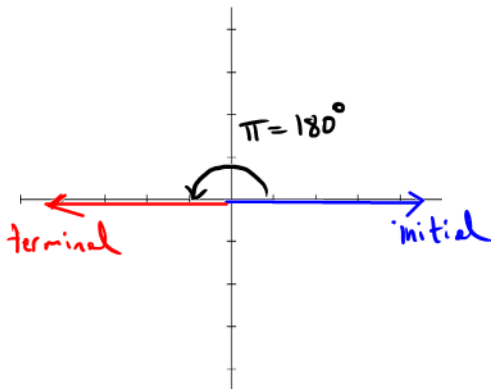
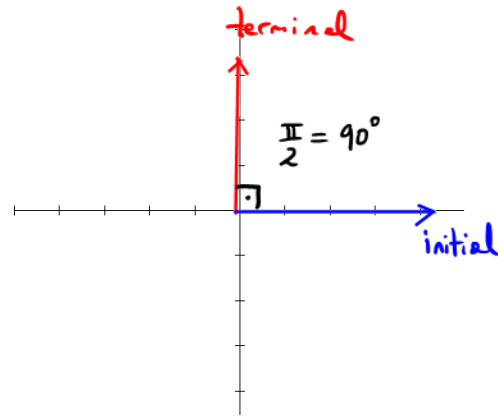
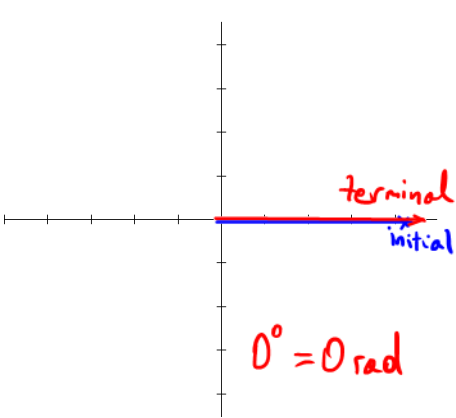
$$\theta_2 = -\frac{15\pi}{8} + 4\pi = -\frac{15\pi}{8} + \frac{32\pi}{8} = \frac{17\pi}{8} (+)$$

$$\theta_3 = -\frac{15\pi}{8} - 2\pi = -\frac{15\pi}{8} - \frac{16\pi}{8} = -\frac{31\pi}{8} (-)$$

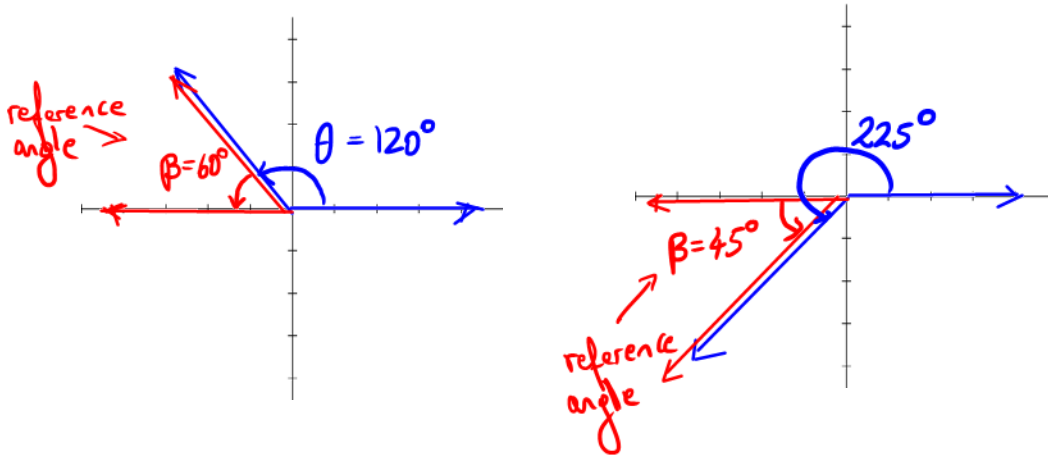
Coterminal

To be continued on Thursday, 03/03.

If an angle is in standard position and its terminal side lies along the x or y axis, then we call the angle a **quadrantal angle**. There are five basic quadrantal angles.



You will need to be able to work with reference angles. Suppose θ is an angle in standard position and θ is not a quadrantal angle. The reference angle for θ is the acute angle of positive measure that is formed by the terminal side of the angle and the x axis.

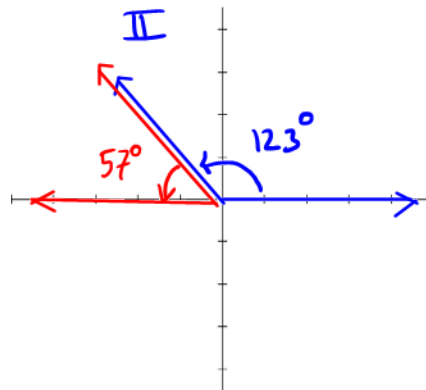


Example 3: Find the reference angle for each of these angles:

Quadrant II

a. 123°

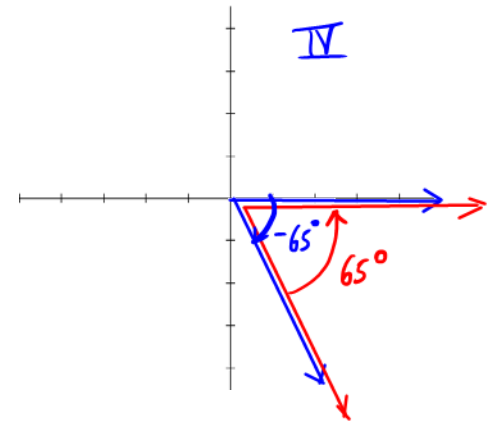
$$180^\circ - 123^\circ = \boxed{57^\circ}$$



Quadrant IV

b. -65°

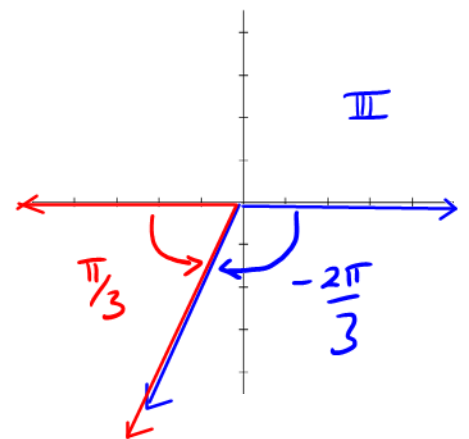
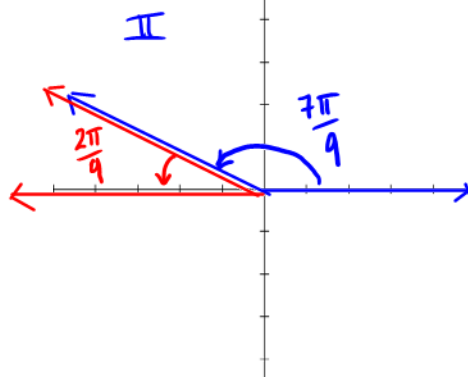
$$-(-65^\circ) = \boxed{65^\circ}$$



Quadrant II

c. $\frac{7\pi}{9}$

$$\pi - \frac{7\pi}{9} = \boxed{\frac{2\pi}{9}}$$



Quadrant III

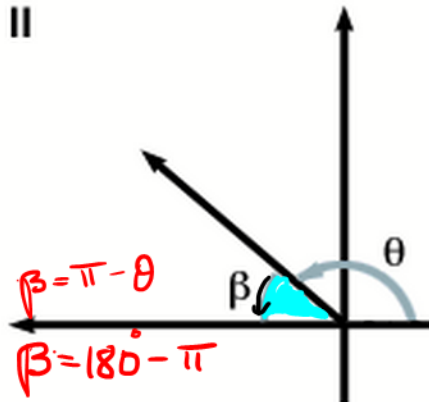
d. $-\frac{2\pi}{3}$

$$\pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$$

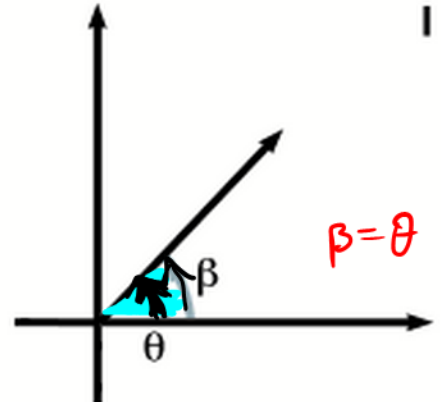
Reference angles: β

Acute
and
positive

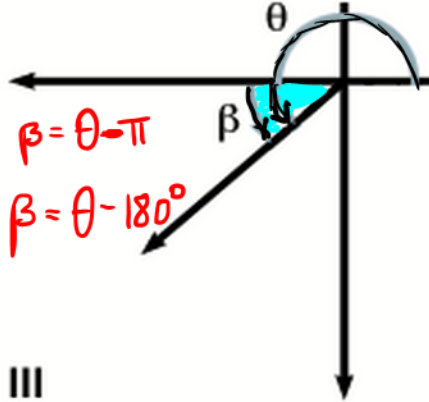
II



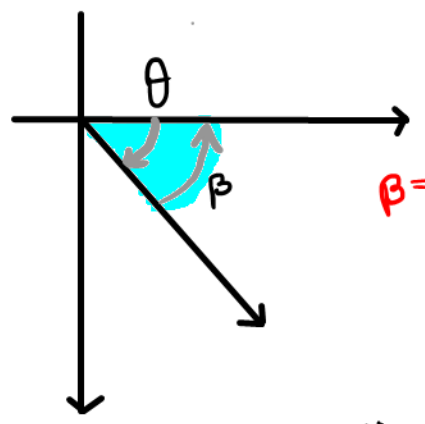
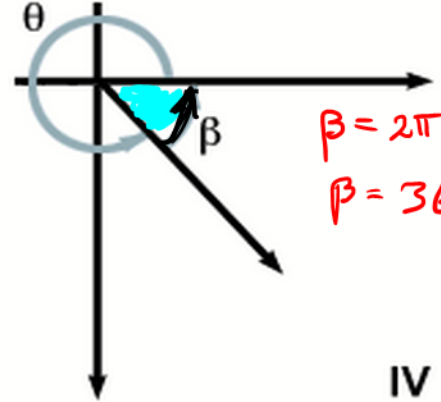
I



III

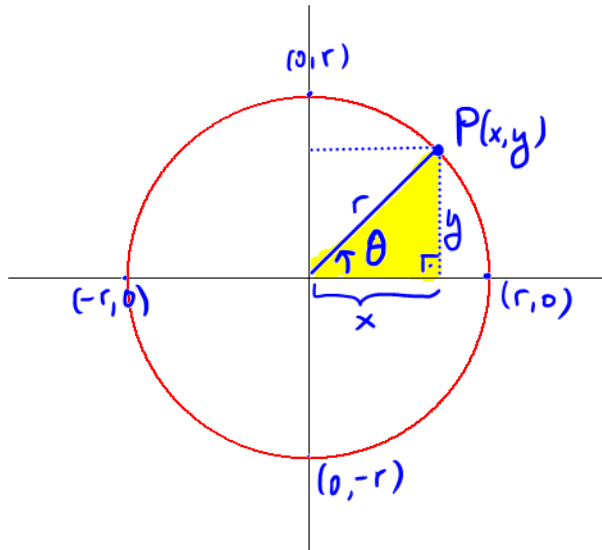


IV



IV

We previously defined the six trigonometric functions of an angle as ratios of the lengths of the sides of a right triangle. Now we will look at them using a circle centered at the origin in the coordinate plane. This circle will have the equation $x^2 + y^2 = r^2$. If we select a point $P(x, y)$ on the circle and draw a ray from the origin through the point, we have created an angle in standard position. The length of the radius will be r .



$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}$$

\Rightarrow If $r = 1$, then

$$\cos \theta = x, \quad \sin \theta = y$$

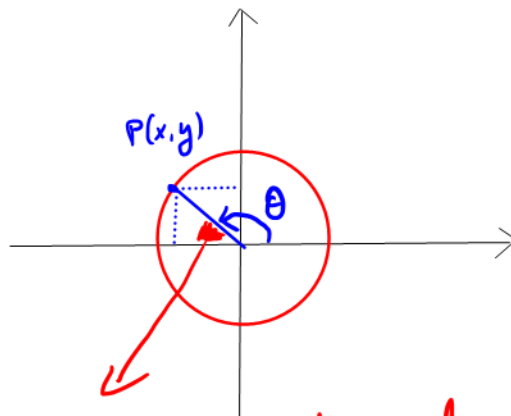
i.e. $x = \cos \theta, \quad y = \sin \theta$

The six trig functions of θ are defined as follows, using the circle above: *Look at the yellow right triangle:*

$\sin \theta = \frac{y}{r}$	$\csc \theta = \frac{r}{y}, y \neq 0$
$\cos \theta = \frac{x}{r}$	$\sec \theta = \frac{r}{x}, x \neq 0$
$\tan \theta = \frac{y}{x}, x \neq 0$	$\cot \theta = \frac{x}{y}, y \neq 0$

where $r^2 = x^2 + y^2$

If θ is a first quadrant angle, these definitions are consistent with the definitions given in Section 4.1.



What if θ is
in 2nd quadrant?
Or 3rd or 4th ???

we look at reference angle!

An **identity** is a statement that is true for all values of the variable. Here are some identities that follow from the definitions above.

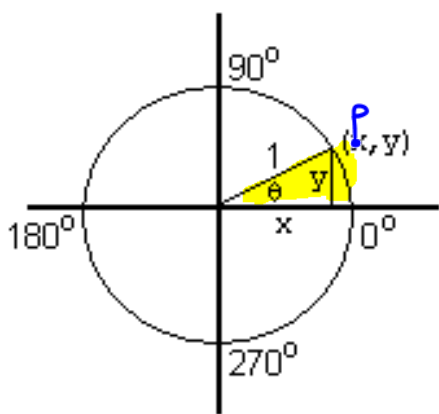
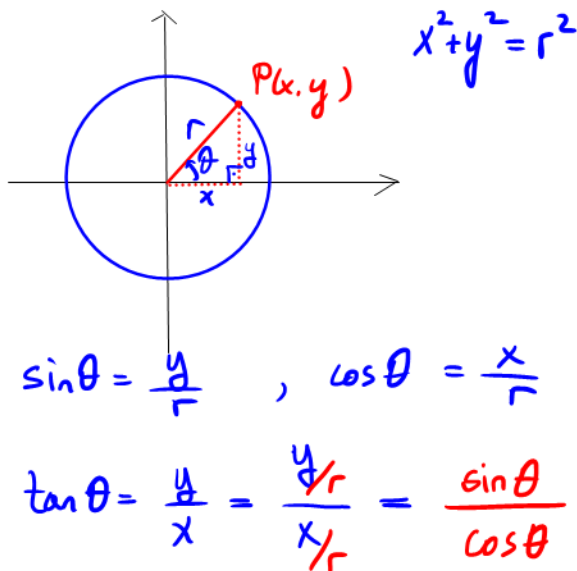
Memorize, they are very important

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$



Every point on unit circle can be identified by trigonometric functions.

We will work most often with a **unit circle**, that is, a circle with radius 1. In this case, each value of r is 1. This adjusts the definitions of the trig functions as follows:

$$\sin \theta = y \quad \csc \theta = \frac{1}{y}, y \neq 0$$

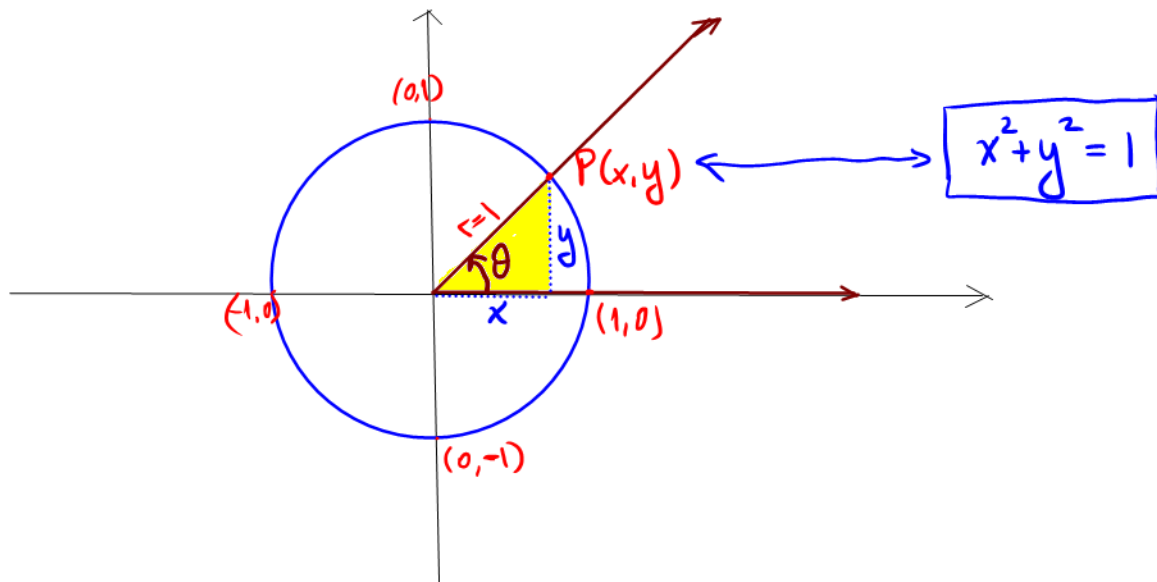
$$\cos \theta = x \quad \sec \theta = \frac{1}{x}, x \neq 0$$

$$\tan \theta = \frac{y}{x}, x \neq 0 \quad \cot \theta = \frac{x}{y}, y \neq 0$$

Look at next page!

We'll get used to the unit circle. ($x^2 + y^2 = 1$, center $(0,0)$, $r=1$)

All trigonometric functions will be given by values of points on unit circle.



Trigonometric functions on unit circle: $r=1$

$$\sin \theta = \frac{\text{Opp}}{\text{Hyp}} = \frac{y}{r} = y$$

$$\cos \theta = \frac{\text{Adj}}{\text{Hyp}} = \frac{x}{r} = x$$

$$\begin{cases} x = \cos \theta \\ y = \sin \theta \end{cases}$$

\Leftrightarrow every point (x,y) on the unit circle

can be expressed

as

$$P(x,y) = P(\cos \theta, \sin \theta)$$

$$\tan \theta = \frac{\text{Opp}}{\text{Adj}} = \frac{y}{x}$$

$$\cot \theta = \frac{\text{Adj}}{\text{Opp}} = \frac{x}{y}$$

$$\sec \theta = \frac{\text{Hyp}}{\text{Adj}} = \frac{1}{x}$$

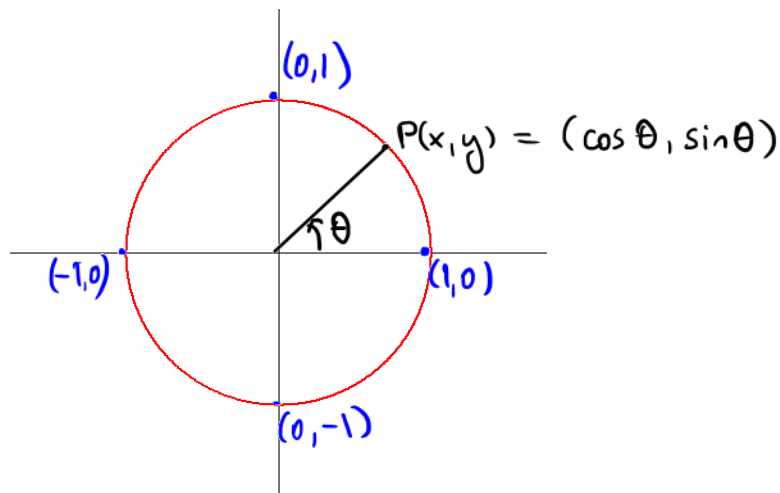
$$\csc \theta = \frac{\text{Hyp}}{\text{Opp}} = \frac{1}{y}$$

Trigonometric Functions of Quadrantal Angles and Special Angles

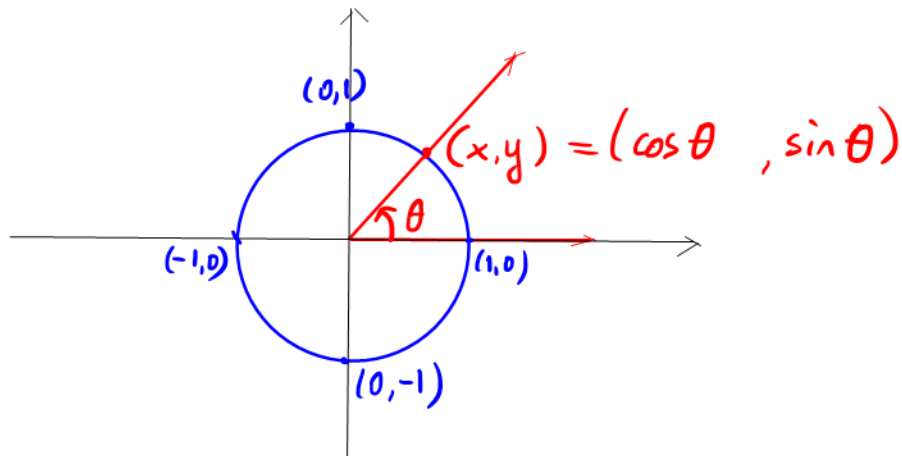
You will need to be able to find the trig functions of quadrantal angles and of angles measuring 30° , 45° or 60° without using a calculator.

Since $\sin \theta = y$ and $\cos \theta = x$, each ordered pair on the unit circle corresponds to $(\cos \theta, \sin \theta)$ of some angle θ .
don't forget

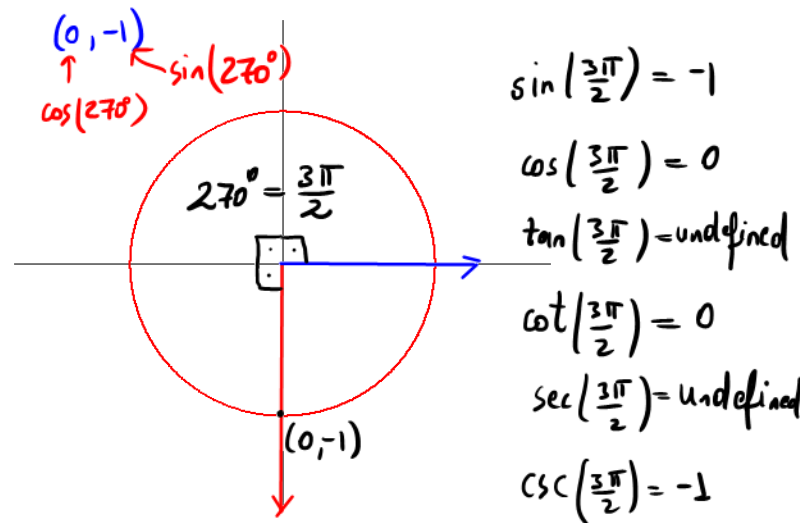
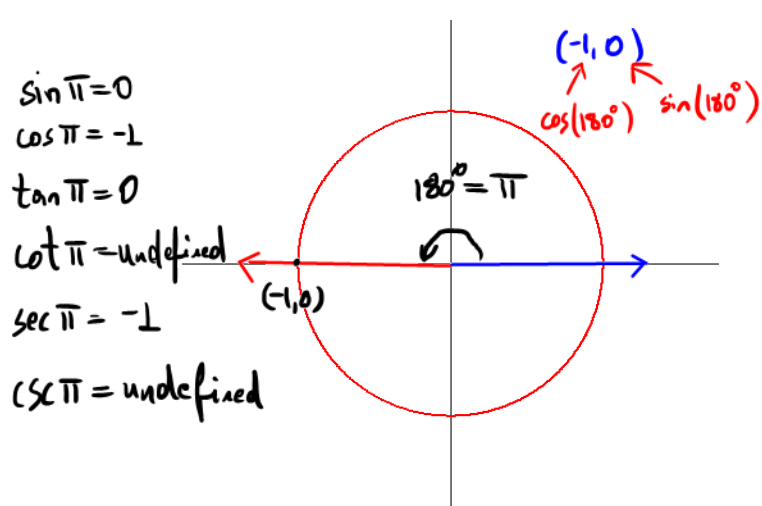
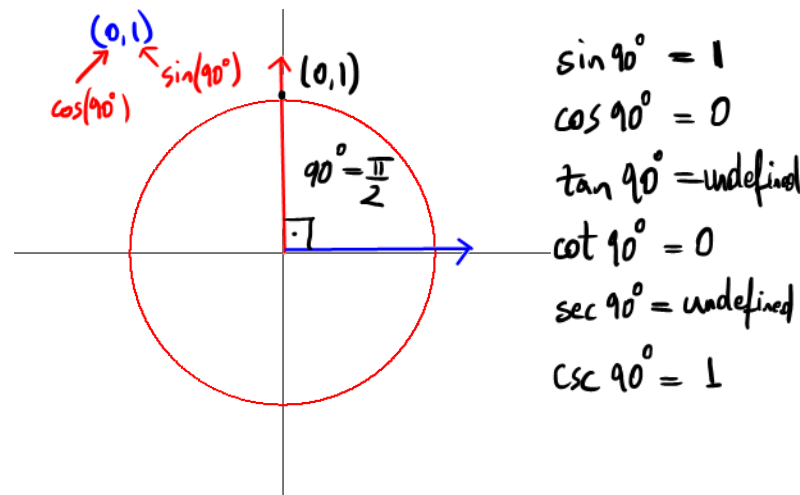
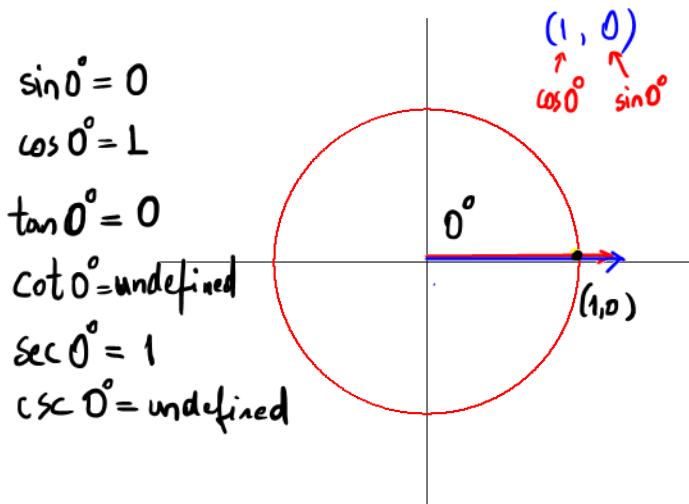
We'll show the values for sine and cosine of the quadrantal angles on this graph. We'll also indicate where the trig functions are positive and where they are negative.



Get used:



Using the identities given above, you can find the other four trig functions of an angle, given just sine and cosine. Note that some values are not defined for quadrantal angles.

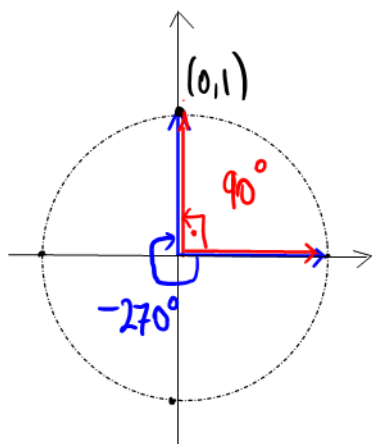


Note that the full angle $360^\circ = 2\pi$ matches the position of the 0° angle. Everything gets repeated.

Values of Trigonometric Functions for Quadrantal Angles

	0°	90°	180°	270°	360°
	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	1	0	-1	0
Cosine	1	0	-1	0	1
Tangent	0	undefined	0	undefined	0
Cotangent	undefined	0	undefined	0	undefined
Secant	1	undefined	-1	undefined	1
Cosecant	undefined	1	undefined	-1	undefined

Example 4: Sketch an angle measuring -270° in the coordinate plane. Then give the six trigonometric functions of the angle. Note that some of the functions may be undefined.



The trigonometric functions for the angle -270° are the same as those for angle 90° :

$$\sin(-270^\circ) = 1$$

$$\cos(-270^\circ) = 0$$

$$\tan(-270^\circ) = \frac{\sin(-270^\circ)}{\cos(-270^\circ)} = \frac{1}{0} \text{ undefined}$$

$$\cot(-270^\circ) = \frac{\cos(-270^\circ)}{\sin(-270^\circ)} = \frac{0}{1} = 0$$

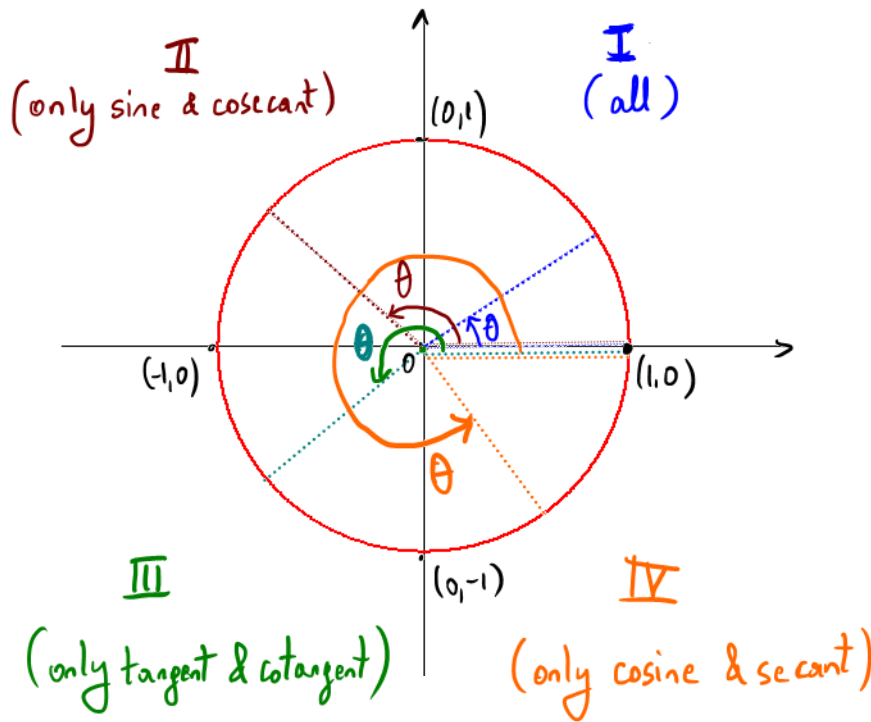
$$\sec(-270^\circ) = \text{undefined}$$

$$\csc(-270^\circ) = 1$$

Recall the signs of the points in each quadrant. Remember, that each point on the unit circle corresponds to an ordered pair, (cosine, sine).

II (-, +)	I (+, +)
III (-, -)	IV (+, -)

$\sin \theta: +$
 $\cos \theta: -$
 $\tan \theta: -$
 $\cot \theta: -$
 $\sec \theta: -$
 $\csc \theta: +$



$\sin \theta: +$
 $\cos \theta: +$
 $\tan \theta: +$
 $\cot \theta: +$
 $\sec \theta: +$
 $\csc \theta: +$

$\sin \theta: -$
 $\cos \theta: -$
 $\tan \theta: +$
 $\cot \theta: +$
 $\sec \theta: -$
 $\csc \theta: -$

$\sin \theta: -$
 $\cos \theta: +$
 $\tan \theta: -$
 $\cot \theta: -$
 $\sec \theta: +$
 $\csc \theta: -$

Example 5: Name the **quadrant** in which both conditions are true:

- II, III I, II
 a. $\cos \theta < 0$ and $\csc \theta > 0$.

$$\csc \theta = \frac{1}{\sin \theta} > 0 \Leftrightarrow \sin \theta > 0 \quad \text{and} \quad \cos \theta < 0 \Leftrightarrow \text{II}$$

- III, IV II, IV
 b. $\sin \theta < 0$ and $\tan \theta < 0$

$$\sin \theta < 0 \quad \& \quad \tan \theta = \frac{\sin \theta}{\cos \theta} < 0 \Rightarrow \cos \theta > 0 \Leftrightarrow \text{IV}$$

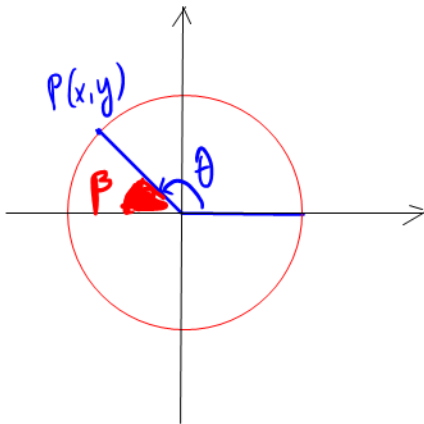
Never forget:

$x \leftrightarrow$ cosine values

$y \leftrightarrow$ sine values

This is a very typical type of problem you'll need to be able to work.

Example 6: Let $P(x, y)$ denote the point where the terminal side of an angle θ intersects the unit circle. If P is in quadrant II and $y = \frac{5}{13}$, find the six trig functions of angle θ .



$$x = \cos \theta \leftarrow \text{negative}$$

$$y = \sin \theta = \frac{5}{13} \leftarrow \text{positive}$$

$$\text{reference} = \beta \Rightarrow \sin \beta = \frac{5}{13}$$

$$\Rightarrow \cos \beta = \frac{12}{13}$$

$$\Rightarrow \sin \theta = \frac{5}{13}$$

$$\cos \theta = -\frac{12}{13}$$

$$\tan \theta = \frac{5/13}{-12/13} = -\frac{5}{12}$$

$$\cot \theta = -\frac{12}{5}$$

$$\sec \theta = \frac{1}{-12/13} = -\frac{13}{12}$$

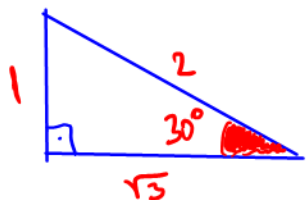
$$\csc \theta = \frac{1}{5/13} = \frac{13}{5}$$

MEMORIZE (the logic)

You'll also need to be able to find the six trig functions of 30° , 60° and 45° angles. YOU MUST KNOW THESE!!!!

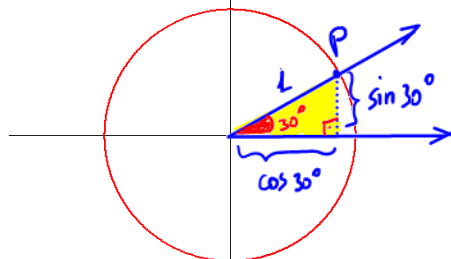
For a 30° angle:

Recall $30^\circ-60^\circ-90^\circ \Delta$:



$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}$$

unit circle



$$\begin{aligned} \text{Point } P &= (\cos 30^\circ, \sin 30^\circ) \\ &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \end{aligned}$$

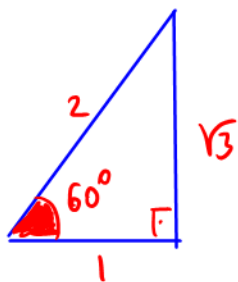
$$\begin{aligned} \sin(30^\circ) &= \frac{1}{2} \\ \cos(30^\circ) &= \frac{\sqrt{3}}{2} \\ \tan(30^\circ) &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \end{aligned}$$

$$\csc(30^\circ) = 2$$

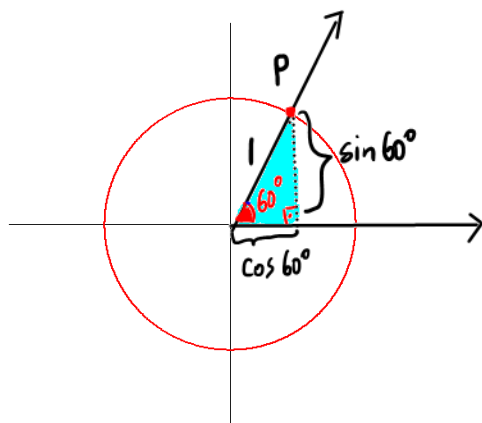
$$\sec(30^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot(30^\circ) = \sqrt{3}$$

For a 60° angle:



$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}$$



$$\begin{aligned} P &= (\cos 60^\circ, \sin 60^\circ) \\ &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{aligned}$$

$$\begin{aligned} \sin(60^\circ) &= \frac{\sqrt{3}}{2} \\ \cos(60^\circ) &= \frac{1}{2} \\ \tan(60^\circ) &= \sqrt{3} \end{aligned}$$

$$\csc(60^\circ) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

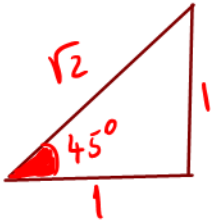
$$\sec(60^\circ) = 2$$

$$\cot(60^\circ) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

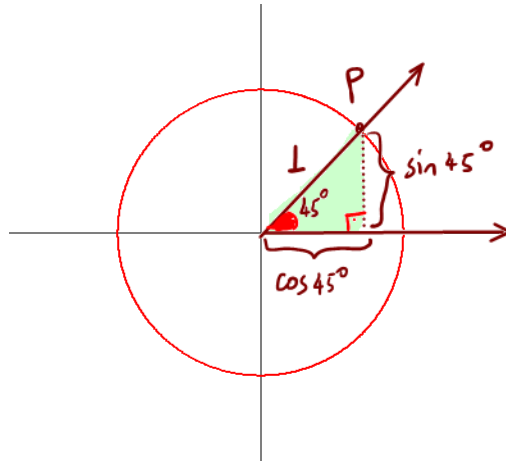
MEMORIZE (the logic)

For a 45° angle:

Recall $45^\circ-45^\circ-90^\circ \Delta$:



$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$$



$$P = (\cos 45^\circ, \sin 45^\circ) \\ = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$\sin(45^\circ) = \frac{\sqrt{2}}{2}$$

$$\cos(45^\circ) = \frac{\sqrt{2}}{2}$$

$$\tan(45^\circ) = 1$$

$$\csc(45^\circ) = \sqrt{2}$$

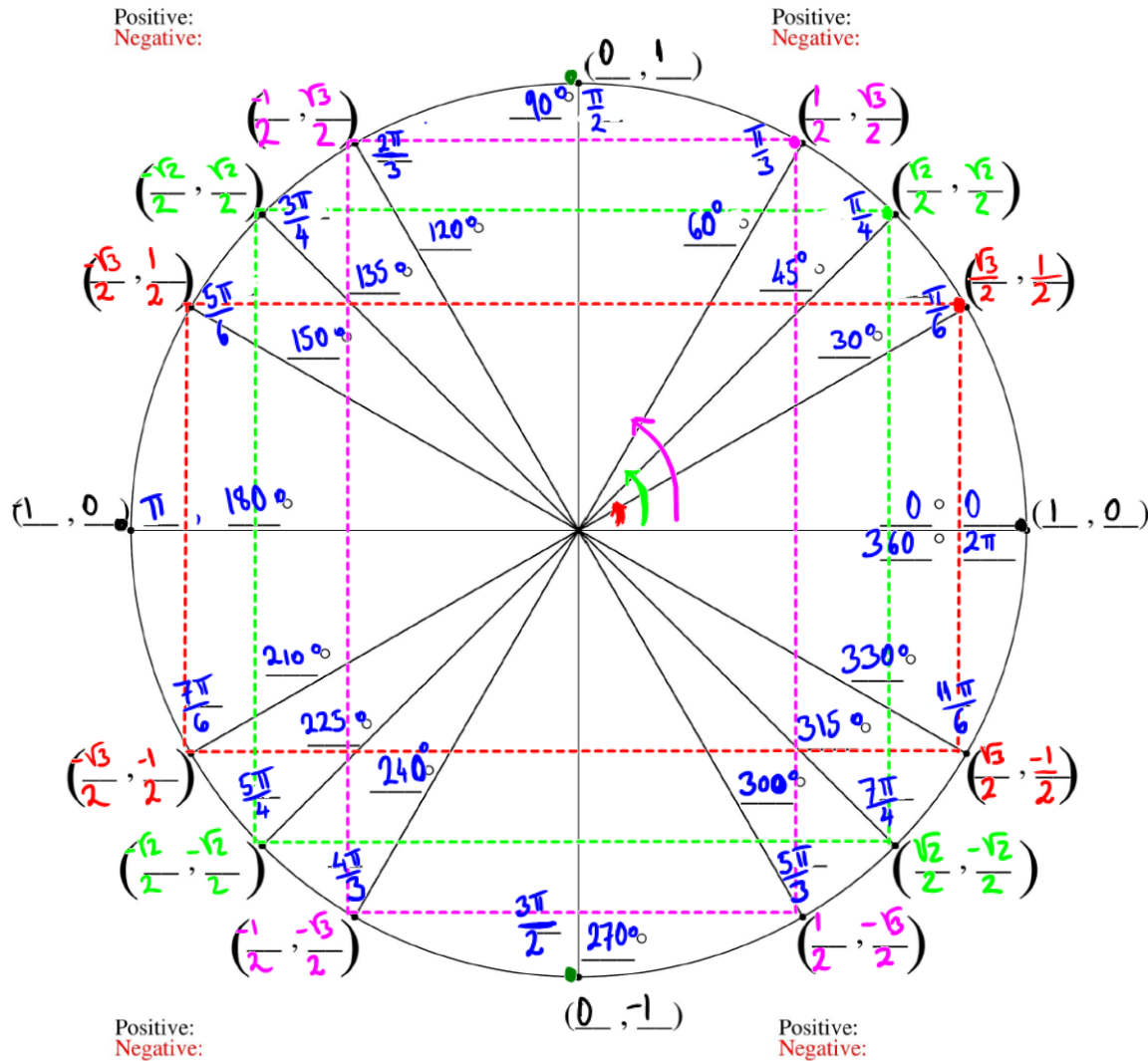
$$\sec(45^\circ) = \sqrt{2}$$

$$\cot(45^\circ) = 1$$

Learn how to complete unit circle

How do we find the trigonometric functions of other special angles?

Method 1: Fill them in. Learn the patterns.



If I ask find $\cos(225^\circ)$, the steps to follow:

- I. locate 225° angle
- II. Find its reference angle
- III. give the answer using the symmetry relationship of angle with its reference!

Method 2: The Chart

Write down the angle measures, starting with 0° and continue until you reach 90° . Under these, write down the equivalent radian measures. Under these, write down the numbers from 0 to 4. Next, take the square root of the values and simplify if possible. Divide each value by 2. This gives you the sine value of each of the angles you need. To find the cosine values, write the previous line in the reverse order.

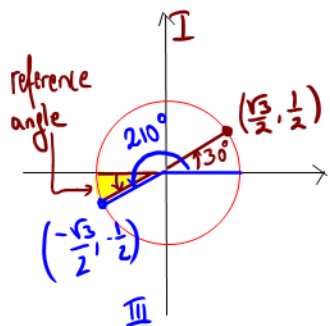
Now you have the sine and cosine values for the quadrantal angles and the special angles. From these, you can find the rest of the trig values for these angles. Write the problem in terms of the reference angle. Then use the chart you created to find the appropriate value.

Angles of Quadrant I .

	0°	30°	45°	60°	90°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined
Cotangent	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
Cosecant	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1

The rest is symmetry
of these values!!!

Example 7: Sketch an angle measuring 210° in the coordinate plane. Give the coordinates of the point where the terminal side of the angle intersects the unit circle. Then state the six trigonometric functions of the angle.



$$\sin(210^\circ) = -\sin(30^\circ) = -\frac{1}{2}$$

$$\cos(210^\circ) = -\cos(30^\circ) = -\frac{\sqrt{3}}{2}$$

$$\tan(210^\circ) = \frac{\sin(210^\circ)}{\cos(210^\circ)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot(210^\circ) = \frac{1}{\tan(210^\circ)} = \sqrt{3}$$

$$\csc(210^\circ) = \frac{1}{\sin(210^\circ)} = -2$$

$$\sec(210^\circ) = \frac{1}{\cos(210^\circ)} = -\frac{2\sqrt{3}}{3}$$

Evaluating Trigonometric Functions Using Reference Angles

1. Determine the reference angle associated with the given angle.
2. Evaluate the given trigonometric function of the reference angle.
3. Affix the appropriate sign determined by the quadrant of the terminal side of the angle in standard position.

Example 8: Evaluate each:

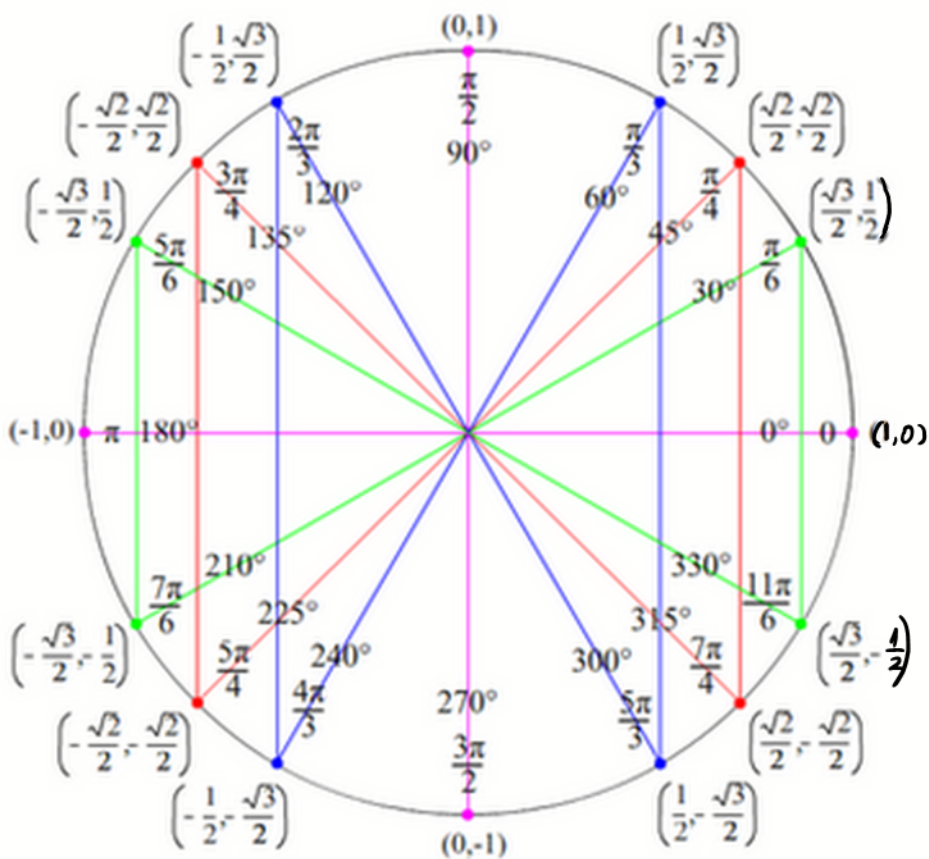
a. $\sin(300^\circ) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2}$

b. $\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)} = -1$

c. $\sec(150^\circ) = \frac{1}{\cos(150^\circ)} = -\frac{2\sqrt{3}}{3}$

d. $\csc\left(\frac{-2\pi}{3}\right) = \frac{1}{\sin\left(\frac{-2\pi}{3}\right)} = -\frac{2\sqrt{3}}{3}$

e. $\sec\left(\frac{11\pi}{6}\right) = \frac{1}{\cos\left(\frac{11\pi}{6}\right)} = \frac{2\sqrt{3}}{3}$

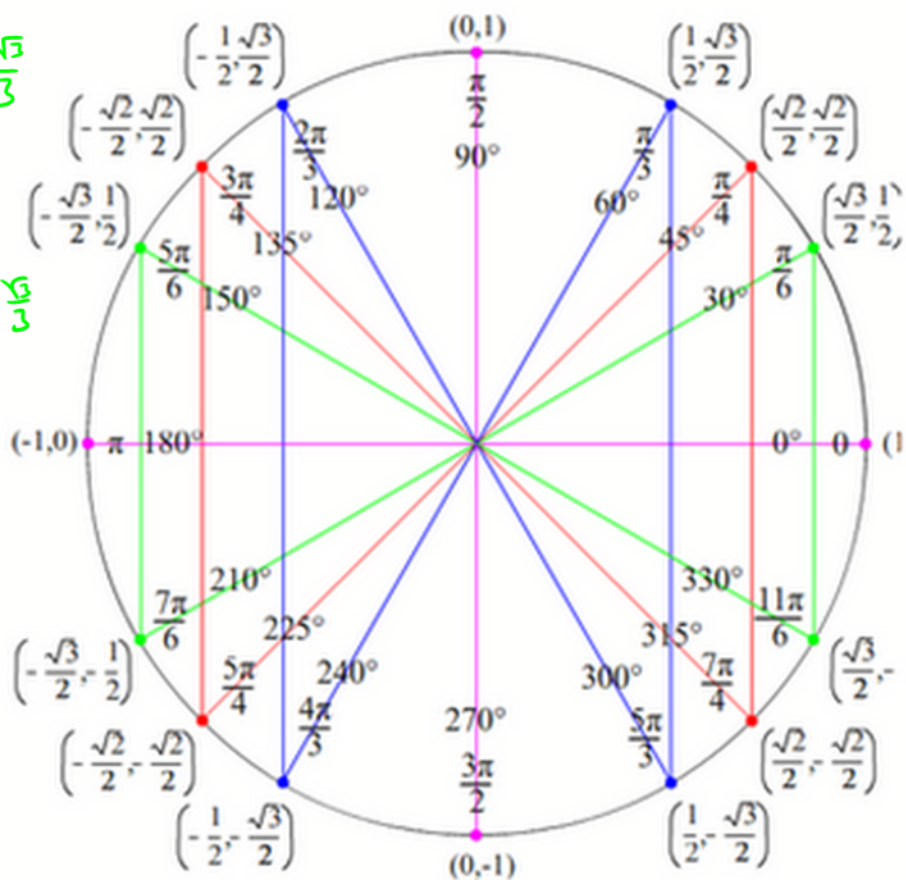


$$f. \tan\left(\frac{7\pi}{6}\right) = \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

$$g. \tan\left(-\frac{5\pi}{6}\right) = \frac{\sin\left(-\frac{5\pi}{6}\right)}{\cos\left(-\frac{5\pi}{6}\right)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

$$h. \tan(240^\circ) = \frac{\sin(240^\circ)}{\cos(240^\circ)} = \sqrt{3}$$

$$i. \cos(-150^\circ) = \cos(150^\circ) = -\frac{\sqrt{3}}{2}$$



STUDY, LEARN, APPLY !!!

