Popper # 13 ← Bubble

$B=3$ $\Longrightarrow$ period $= \frac{2\pi}{B} = \frac{2\pi}{3}$

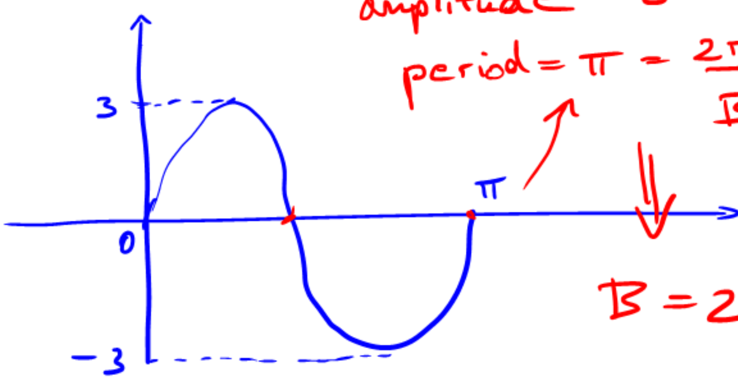
① given $f(x) = 2\cos(3x) - 5$, find period look for $B$

A. $2\pi$  B. $\pi$  C. $\frac{2\pi}{3}$  D. $\frac{\pi}{3}$

amplitude $= 3 \Longrightarrow A = 3$

period $= \pi = \frac{2\pi}{B}$

②




$B = 2$

A. $f(x) = 3\sin x$

B. $f(x) = 3\sin(2x)$

C. $f(x) = \sin(2x)$

D. $f(x) = \sin x$

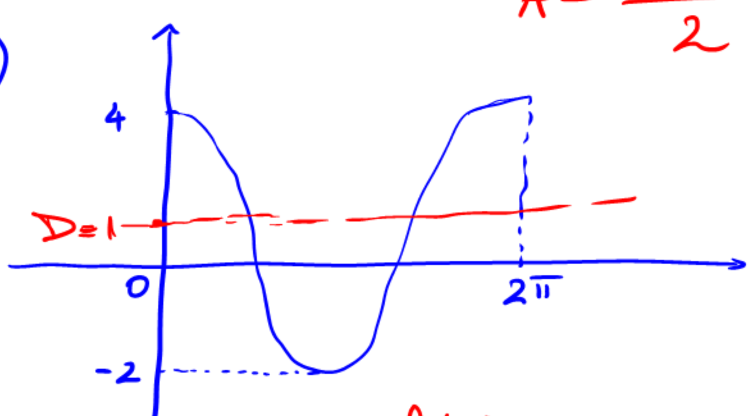
$\Longrightarrow f(x) = 3\sin(2x)$

③ $f(x) = 4\cos\left(x - \frac{\pi}{2}\right) + 2$ $\Longrightarrow$ phase shift $= ?$ $\frac{C}{B} = \frac{\pi/2}{1}$

A. $\frac{\pi}{2}$  B. $\frac{\pi}{8}$  C. $2$  D. none

$A = \frac{4-(-2)}{2} = 3$  $D = \frac{4+(-2)}{2} = 1$

④




A. $f(x) = 3\cos x$

B. $f(x) = 3\cos x + 1$

C. $f(x) = \cos x + 1$

D. $f(x) = \sin x$.

$f(x) = 3\cos x + 1$

**Section 5.2 - Graphs of the Sine and Cosine Functions**

In this section, we will graph the basic sine function and the basic cosine function and then graph other sine and cosine functions using transformations. Much of what we will do in graphing these problems will be the same as earlier graphing using transformations.

**Definition:** A non-constant function $f$ is said to be periodic if there is a number $p > 0$ such that $f(x+p) = f(x)$ for all $x$ in the domain of $f$. The smallest such number $p$ is called the **period** of $f$.

period – interval where f repeats itself.

The graphs of periodic functions display patterns that repeat themselves at regular intervals.

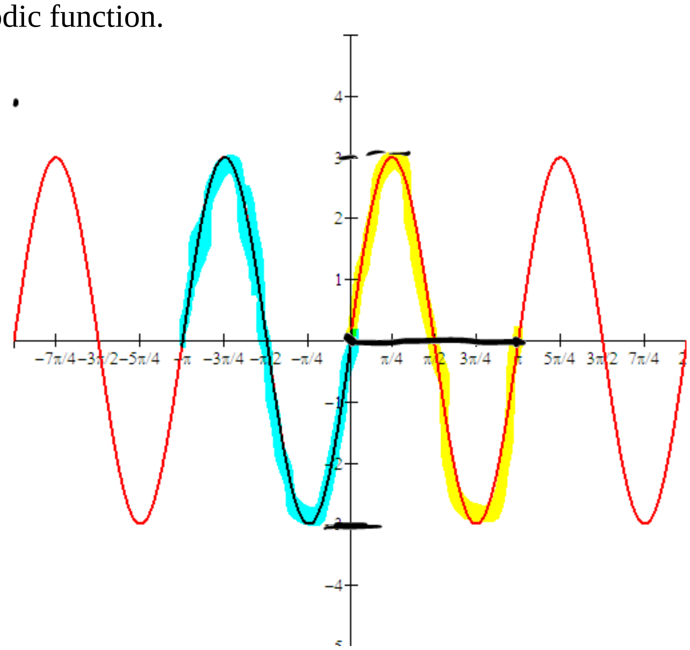
For sine and cosine functions $p = 2\pi$.

**Definition:** For a periodic function $f$ with maximum value $M$ and minimum value $m$.

The **amplitude** of the function is: $\frac{M-m}{2}$. = half of the distance between maximum and minimum.

In other words the amplitude is half the height.

**Example 1:** State the period and amplitude of the periodic function.



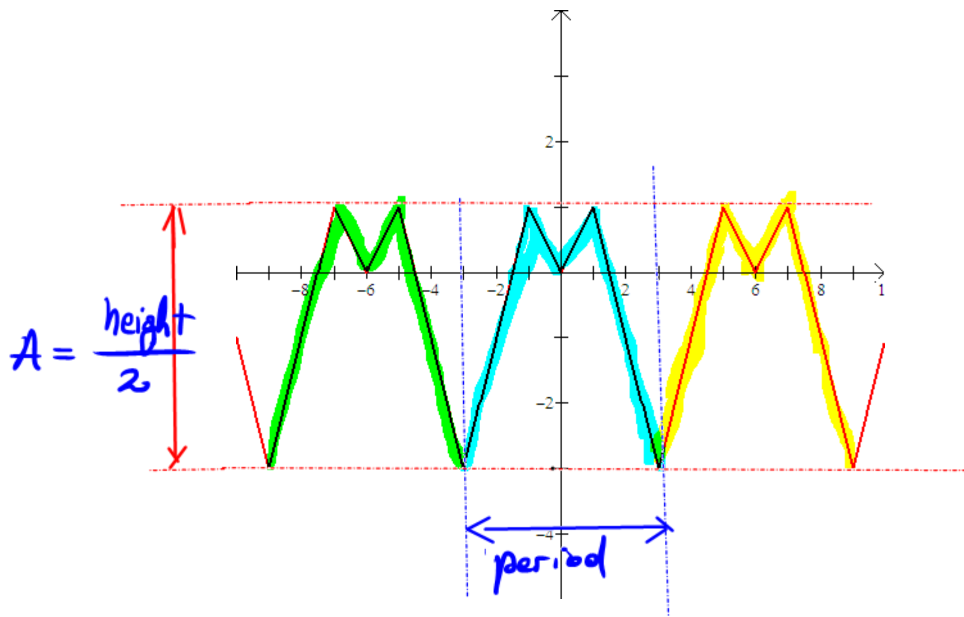

period $= \pi$

Amplitude $= 3 = \frac{6}{2}$

**Example 2:** State the period and amplitude of the periodic function:

period = 6

amplitude $= \frac{4}{2} = 2$




**Note:** For a periodic function *f*, the period of the graph is the length of the interval needed to draw one complete cycle of the graph. For a basic sine or cosine function, the period is $2\pi$.

For a basic sine or cosine function, the maximum value is 1 and the minimum value is -1, so the **amplitude** is $\frac{1-(-1)}{2} = 1$.

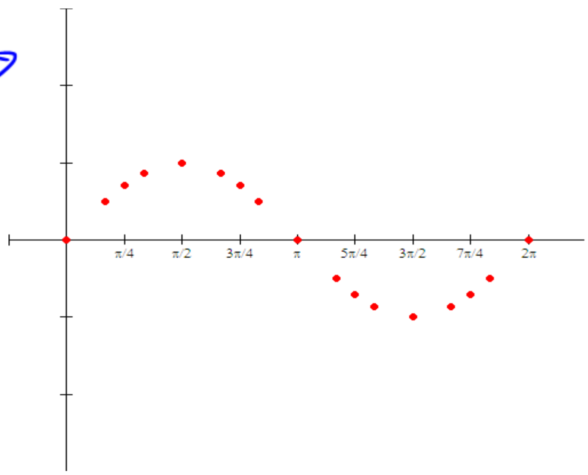
We'll start with the graph of the basic sine function, $f(x) = \sin(x)$. The domain of this function is $(-\infty, \infty)$ and the range is [-1, 1]. We typically graph just one complete period of the graph, that is on the interval $[0, 2\pi]$.

We'll make a table of values:

| ***x*** | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **sin *x*** | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 |

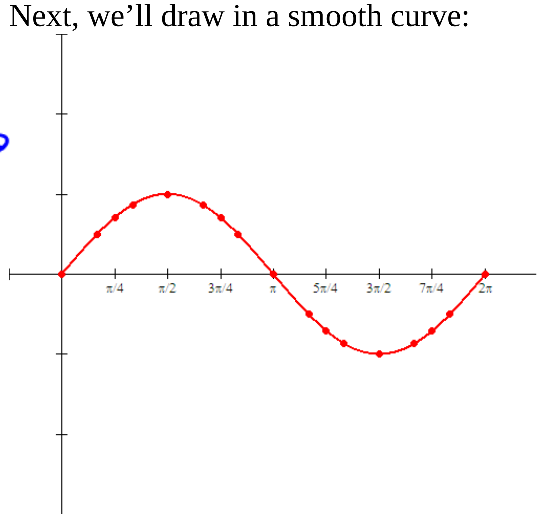
Plot

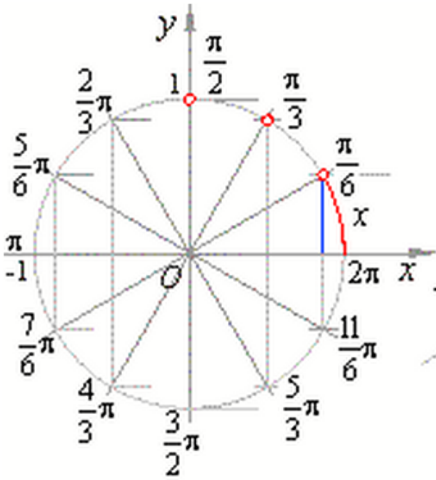
Then using these ordered pairs, we can sketch a graph of the function.

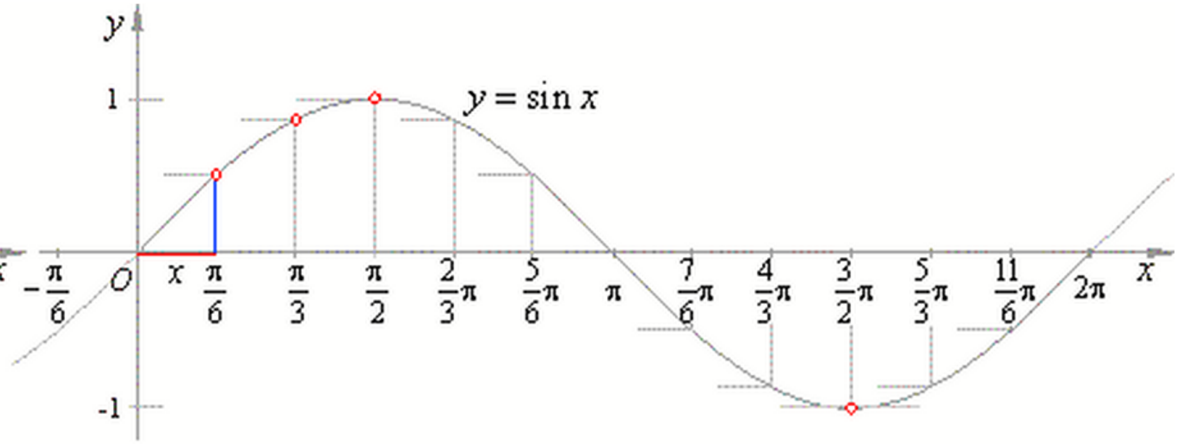


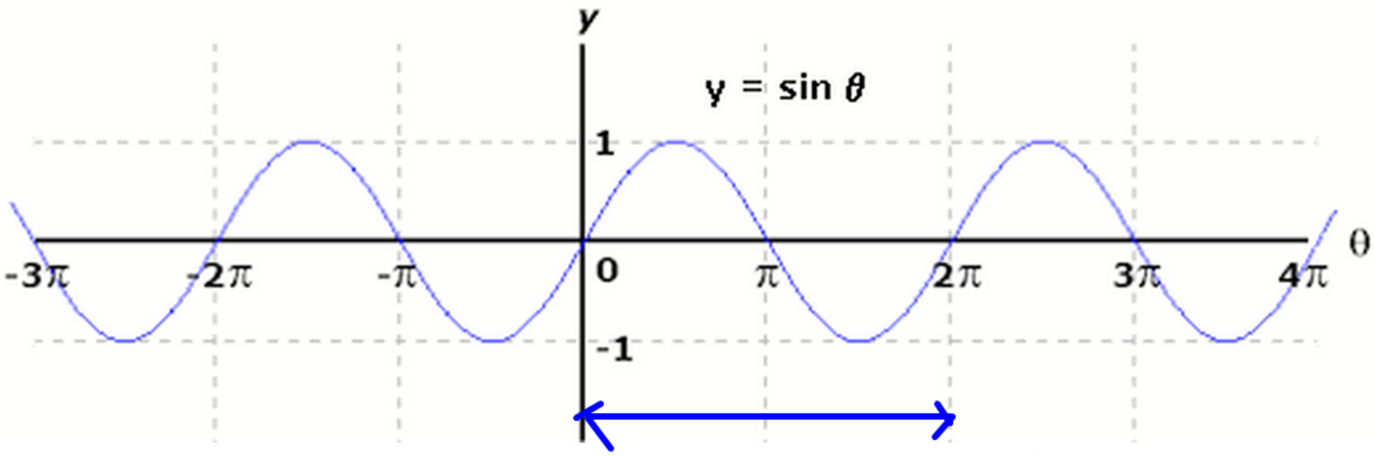

Connect to form a smooth curve.

Next, we'll draw in a smooth curve:





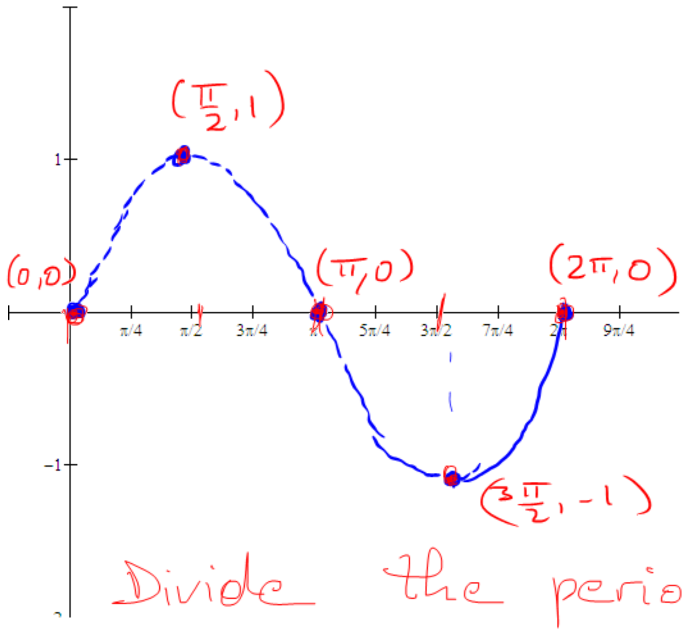

$2\pi$ = period of sine function.

$f(x) = \sin x$

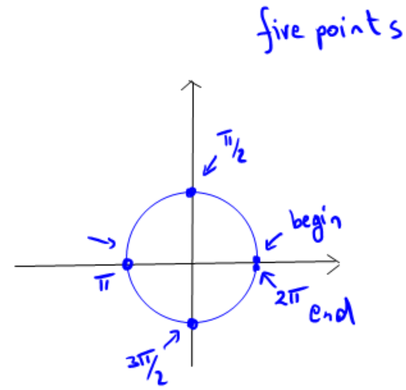
Period $= 2\pi$

Amplitude $= 1$

Drawing all of these points is rather tedious. We'll ask you to learn the shape of the graph and just graph five basic points, the $x$ and $y$ intercepts and the maximum and the minimum.

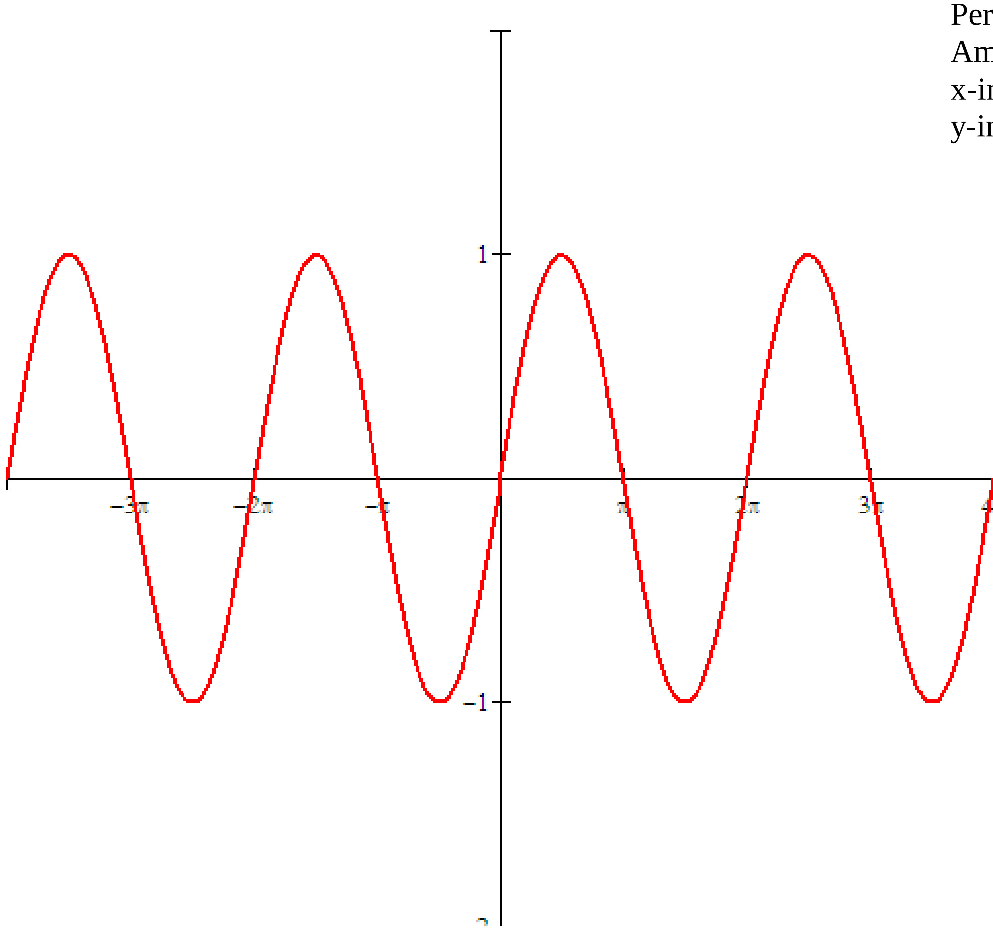



Period: $2\pi$
Amplitude: 1
x-intercepts: $\pi, 2\pi$
y-intercept: (0,0)




Divide the period interval into four equal pieces.

Big picture: $f(x) = \sin(x)$




Period: $2\pi$
Amplitude: 1
x-intercepts: $k\pi$
y-intercept: (0,0)

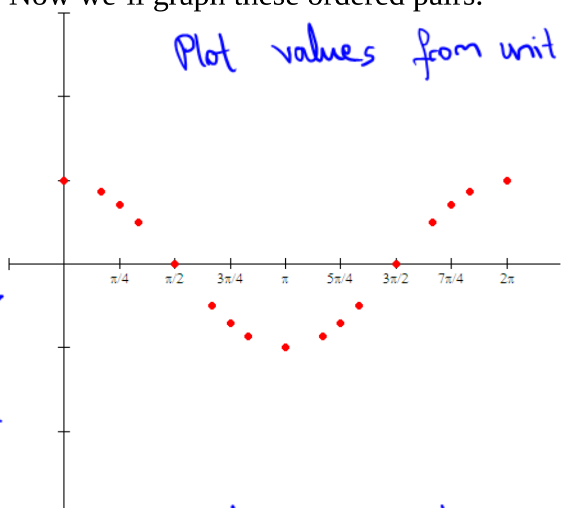
Now we'll repeat the process for the basic cosine function $f(x) = \cos(x)$. The domain of this function is $(-\infty, \infty)$ and the range is [-1, 1]. Again, we typically graph just one complete period of the graph, that is on the interval $[0, 2\pi]$.

Here is the table of values for $f(x) = \cos(x)$:

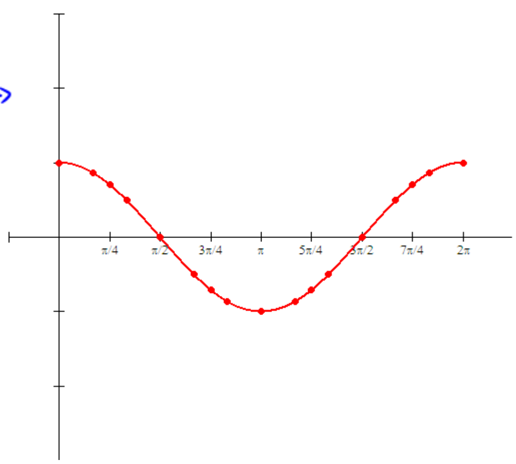
| **x** | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **cos x** | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |

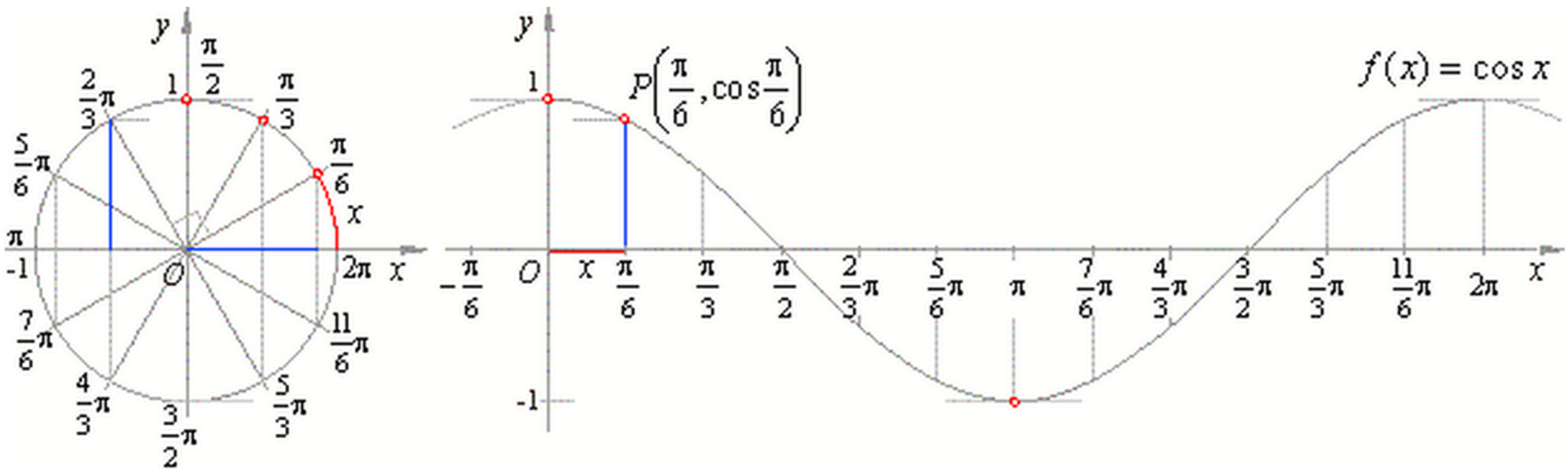
Now we'll graph these ordered pairs.

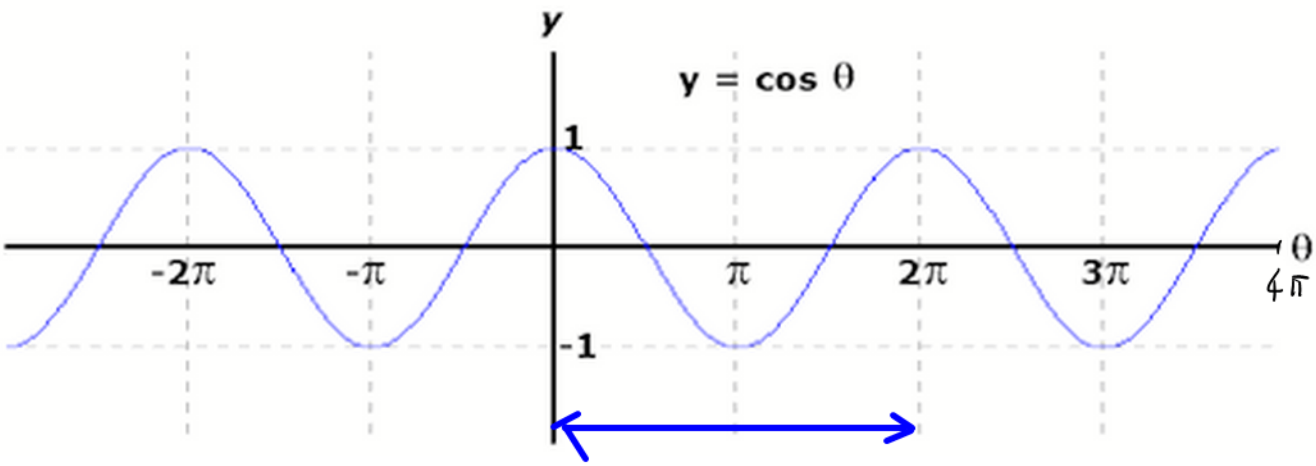
Plot values from unit circle




Connect the dots to get a smooth curve.





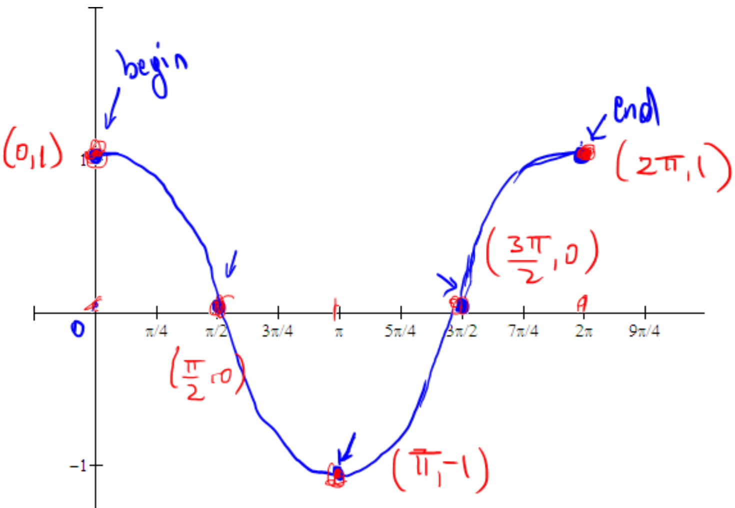

$2\pi =$ period of cosine function.

$f(x) = \cos x,$

Period $= 2\pi$.

Amplitude $= 1$

For the basic cosine graph, you'll need to remember the basic shape and graph the *x* and *y* intercepts as well as the maximum and minimum points. Five points

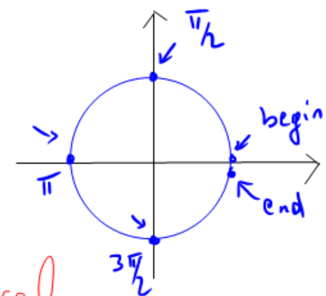



Period: $2\pi$

Amplitude: 1

x-intercepts: $\frac{\pi}{2}, \frac{3\pi}{2}$

y-intercept: (0,1)




Divide the period interval into four equal pieces.

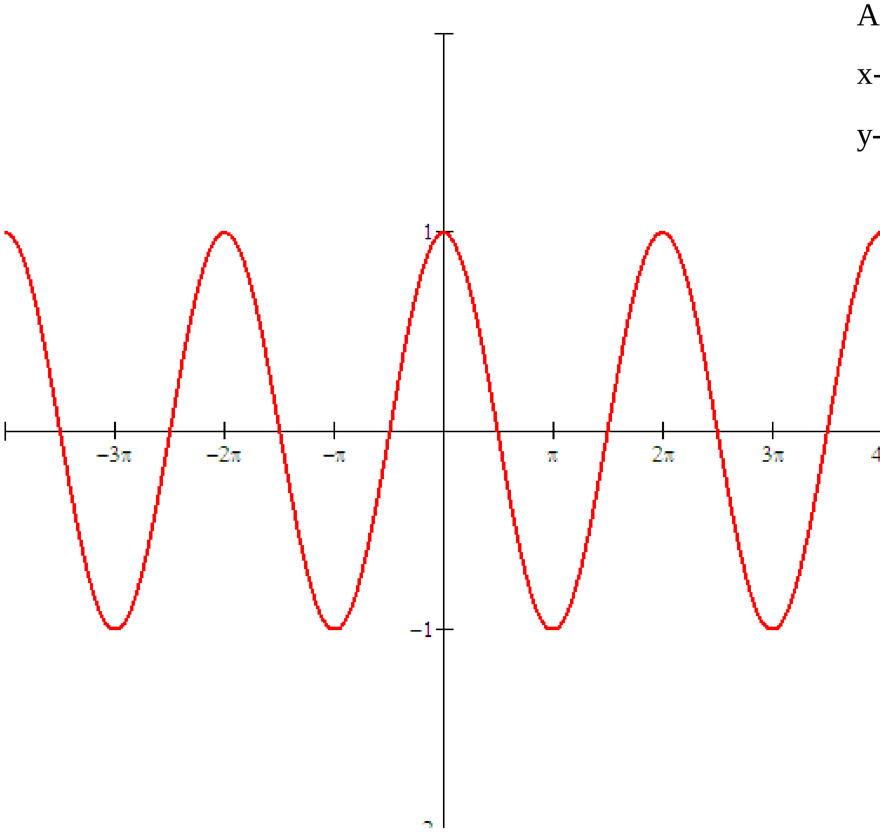
Big picture: $f(x) = \cos(x)$

Period: $2\pi$

Amplitude: 1

x-intercepts: $\frac{k\pi}{2}$ ( k is an odd integer)

y-intercept: (0,1)



→ Review the Transformations' Techniques for any function

Now we'll turn our attention to transformations of the basic sine and cosine functions. These functions will be of the form $f(x) = A\sin(Bx - C) + D$ or $g(x) = A\cos(Bx - C) + D$. We can stretch or shrink sine and cosine functions, both vertically and horizontally. We can reflect them about the *x* axis, the *y* axis or both axes, and we can translate the graphs either vertically, horizontally or both. Next we'll see how the values for *A*, *B*, *C* and *D* affect the graph of the sine or cosine function.

$f(x) = \sin x \leftarrow$ shape $\qquad g(x) = \cos x \leftarrow$ shape

**Graphing** $f(x) = A\sin(Bx - C) + D$ or $g(x) = A\cos(Bx - C) + D$ $= A\cos\left[B\left(x - \frac{C}{B}\right)\right] + D$

→ apply transformations ←

Rewrite:

$f(x) = A\sin\left[B\left(x - \frac{C}{B}\right)\right] + D$

↓ horizontal shift, we call phase shift

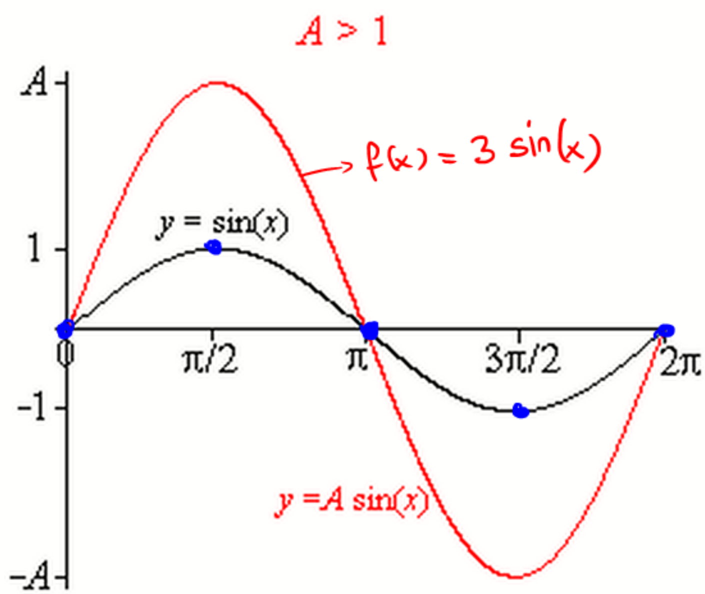
- The **amplitude** of the graph of is $|A|$.
- The period of the function is: $\dfrac{2\pi}{B}$.
- If *A* > 1, this will **stretch** the graph vertically.
  0 < *A* < 1, this will **shrink** the graph vertically
  If *A* < 0, the graph will be a reflection about the *x* axis.
- If B >1 , this will **shrink the graph horizontally by a factor of 1/B.**
  If 0<B<1, this will **stretch the graph horizontally** by a factor of 1/B.
- **Vertical Shift:** Shift the original graph *D* units UP if *D* > 0, *D* units DOWN if *D* < 0.
- **Phase shift:** The function will be shifted $\dfrac{C}{B}$ units to the right if $\dfrac{C}{B} > 0$ and to the left if $\dfrac{C}{B} < 0$. The number $\dfrac{C}{B}$ is called the **phase shift**.

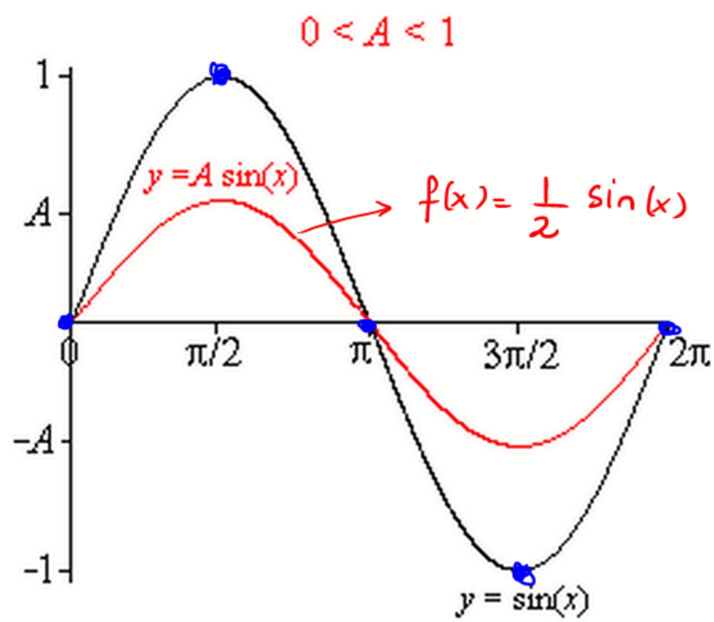
Note: **Horizontal Shift:** If the function is of the form $f(x) = \sin(x - C)$ or $f(x) = \cos(x - C)$, then shift the original graph *C* units to the RIGHT if *C* > 0 and *C* units to the LEFT if *C* < 0.

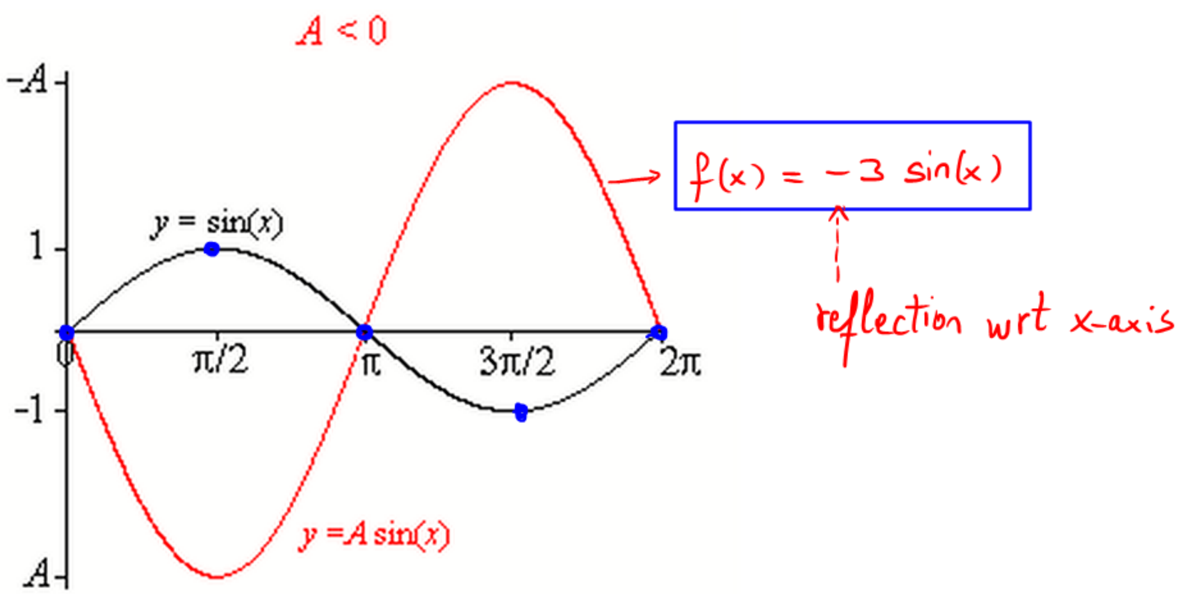
Vertical Stretching (A>1) or Vertical Shrinking (0<A<1): f(x) = A sin(x)

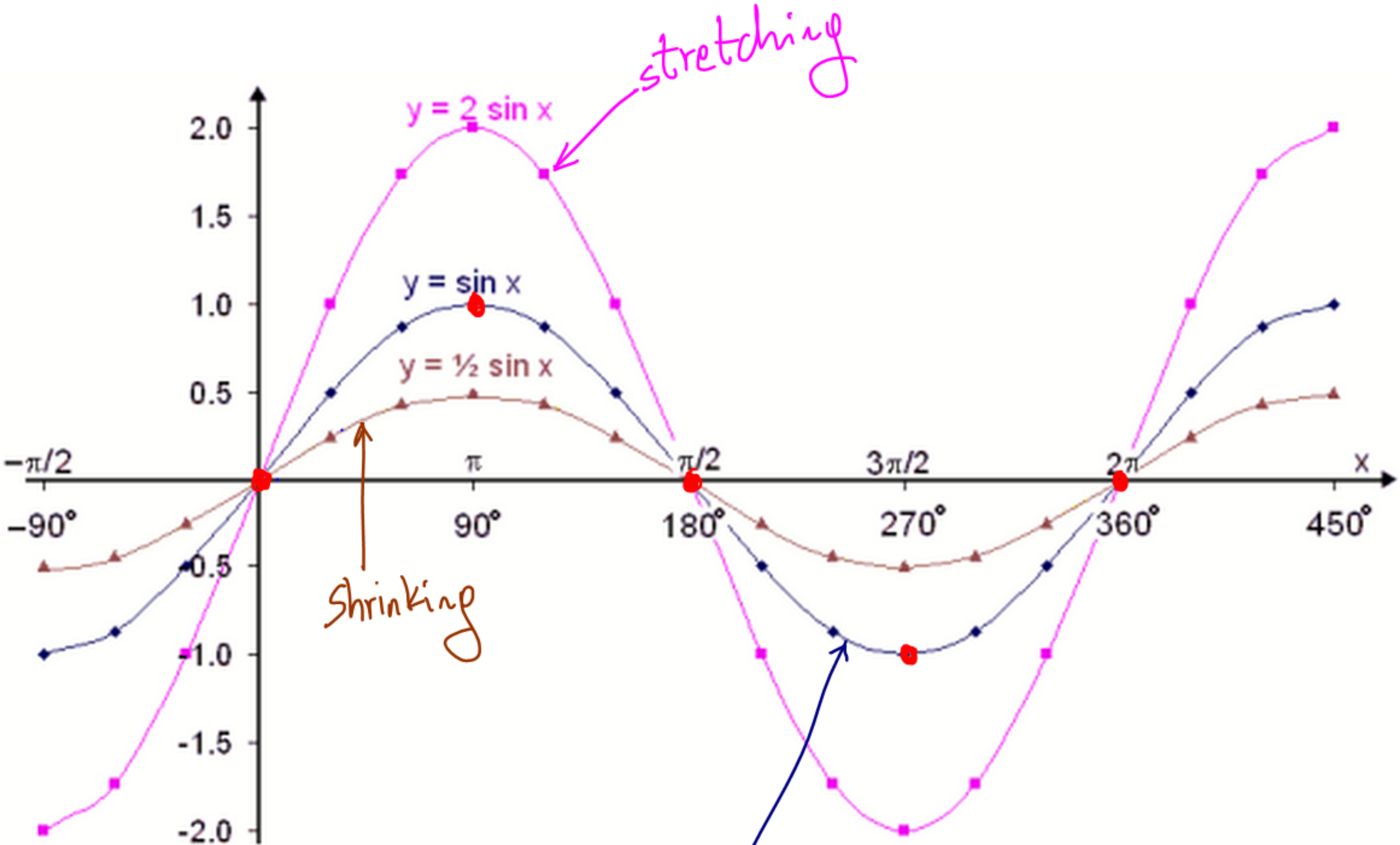
or
$f(x) = A\cos(x)$

If A > 1, this will stretch the graph vertically.
0 < A < 1, this will shrink the graph vertically
If A < 0, the graph will be a reflection about the x axis.






normal basic curve

$f(x) = \sin x$

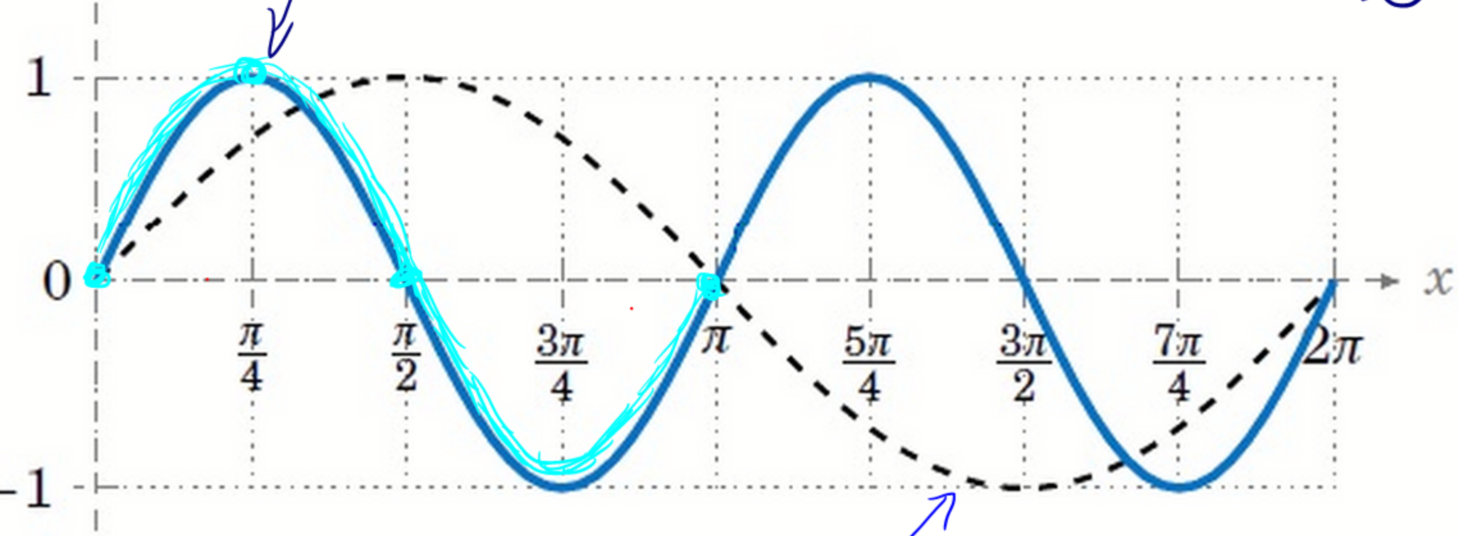
Do not forget the

five points.

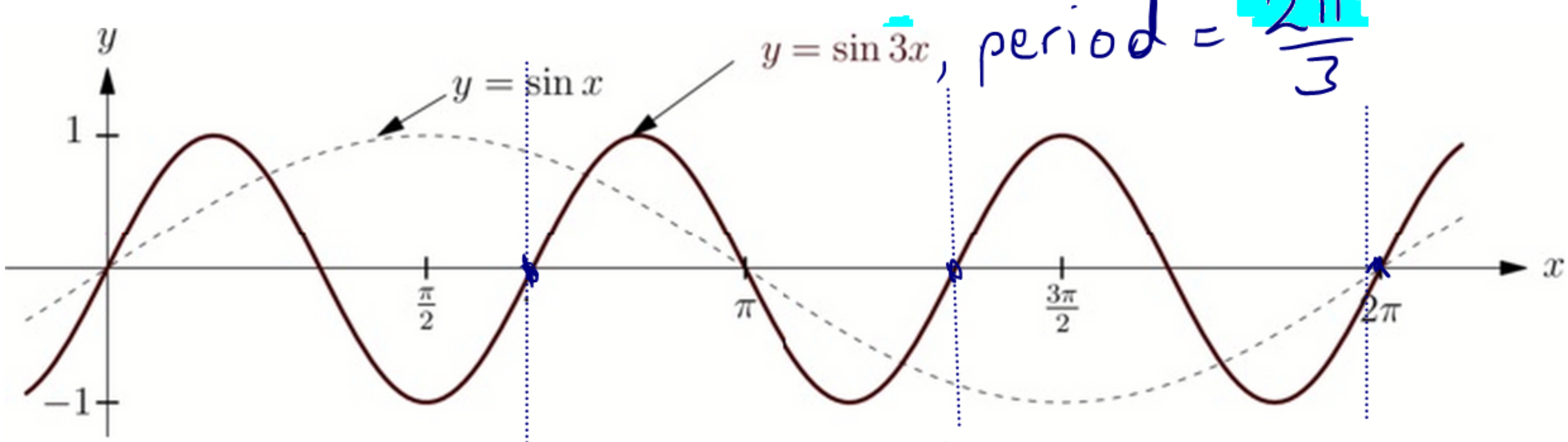
Horizontal Shrinking (B>1) or Horizontal Stretching (0<B<1): f(x)=sin(Bx)

If B >1 , this will shrink the graph horizontally by a factor of 1/B.
If 0<B<1, this will stretch the graph horizontally by a factor of 1/B.

$f(x) = \sin(2x)$ , period $= \pi = \frac{2\pi}{2}$

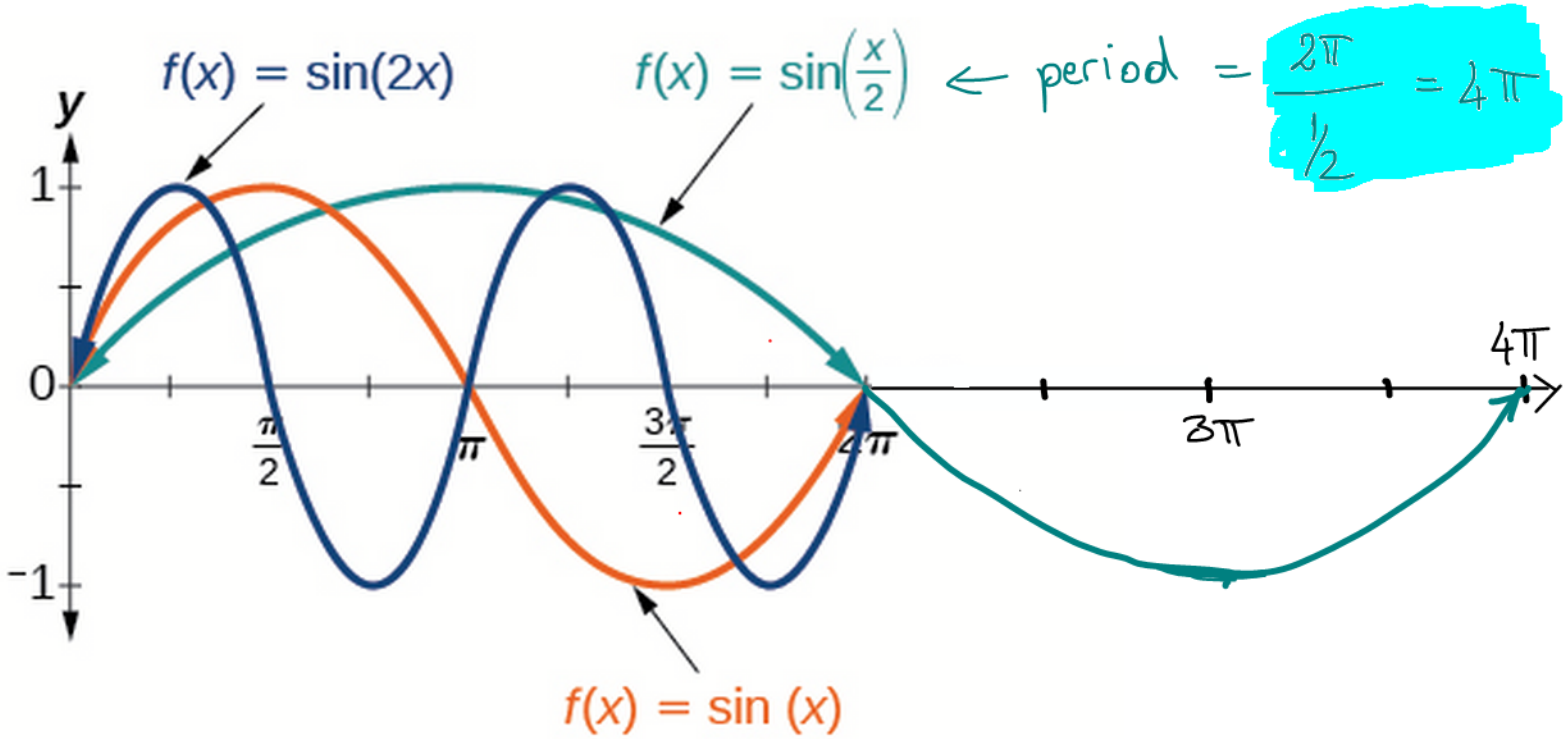



$y = \sin x$ , period $= 2\pi$

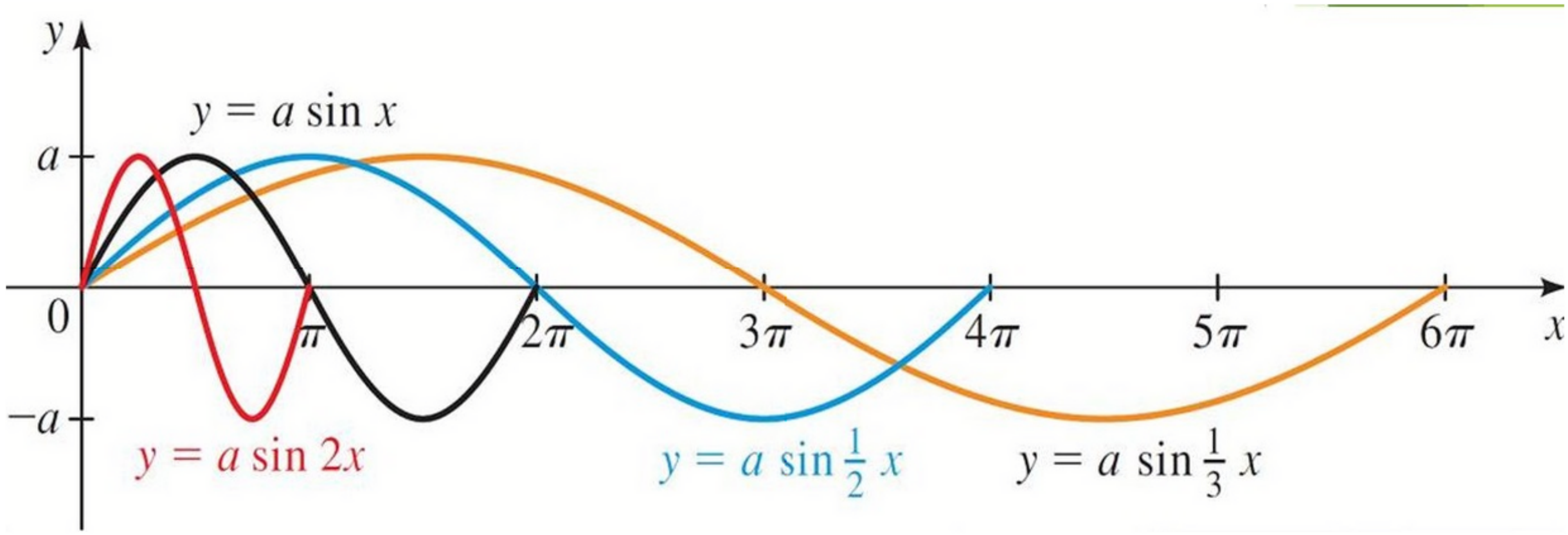



, period $= \frac{2\pi}{3}$

$2\pi/3$ $2\pi/3$ $2\pi/3$



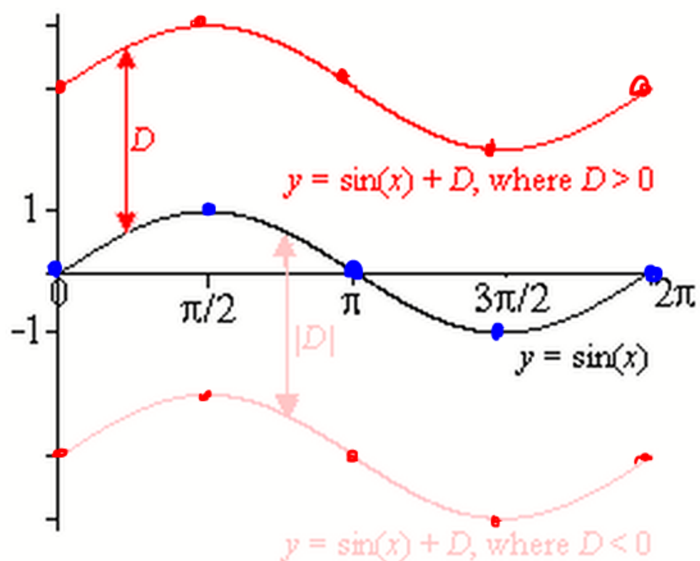
f(x) = sin(2x)
f(x) = sin(x/2)
← period = 2π / ½ = 4π
y
1
0
−1
π/2
π
3π/2
2π
3π
4π
f(x) = sin (x)

y
y = a sin x
a
0
−a
π
2π
3π
4π
5π
6π
x
y = a sin 2x
y = a sin ½ x
y = a sin ⅓ x

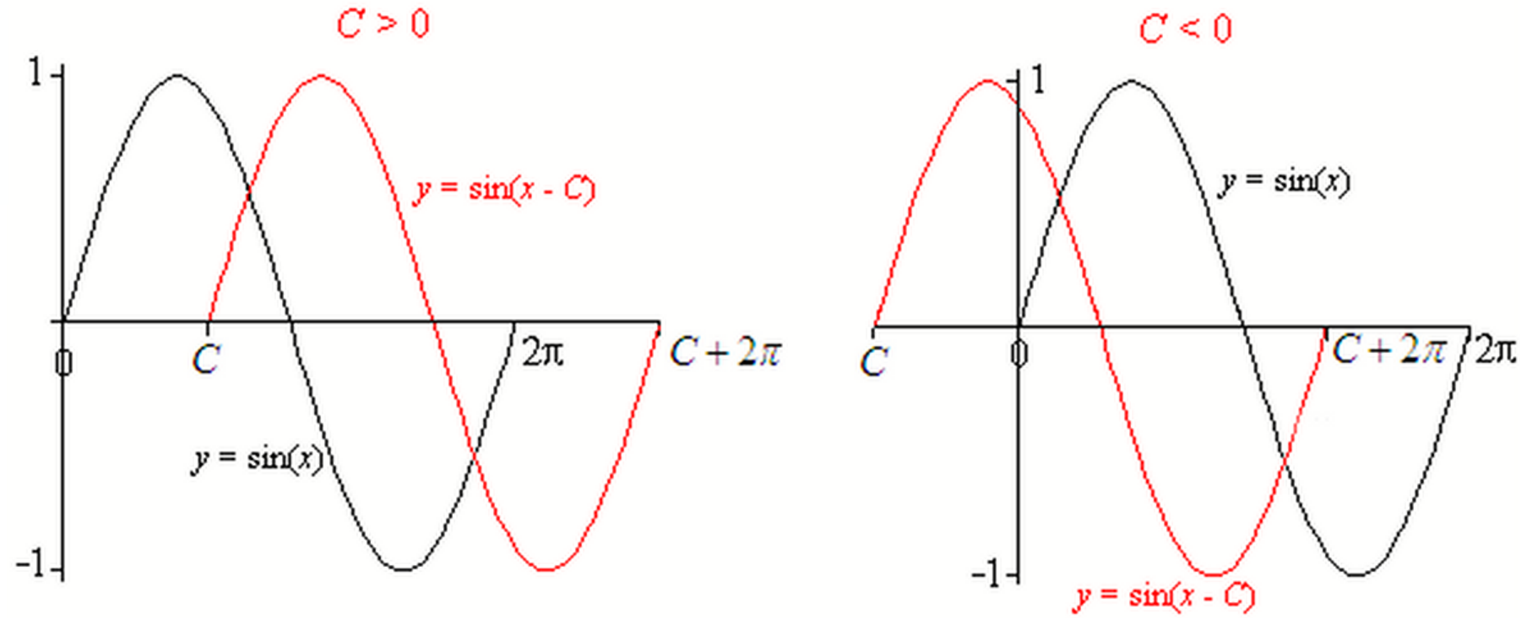
Vertical Shift: $f(x) = \sin(x) + D$ or $f(x) = \cos x + D$

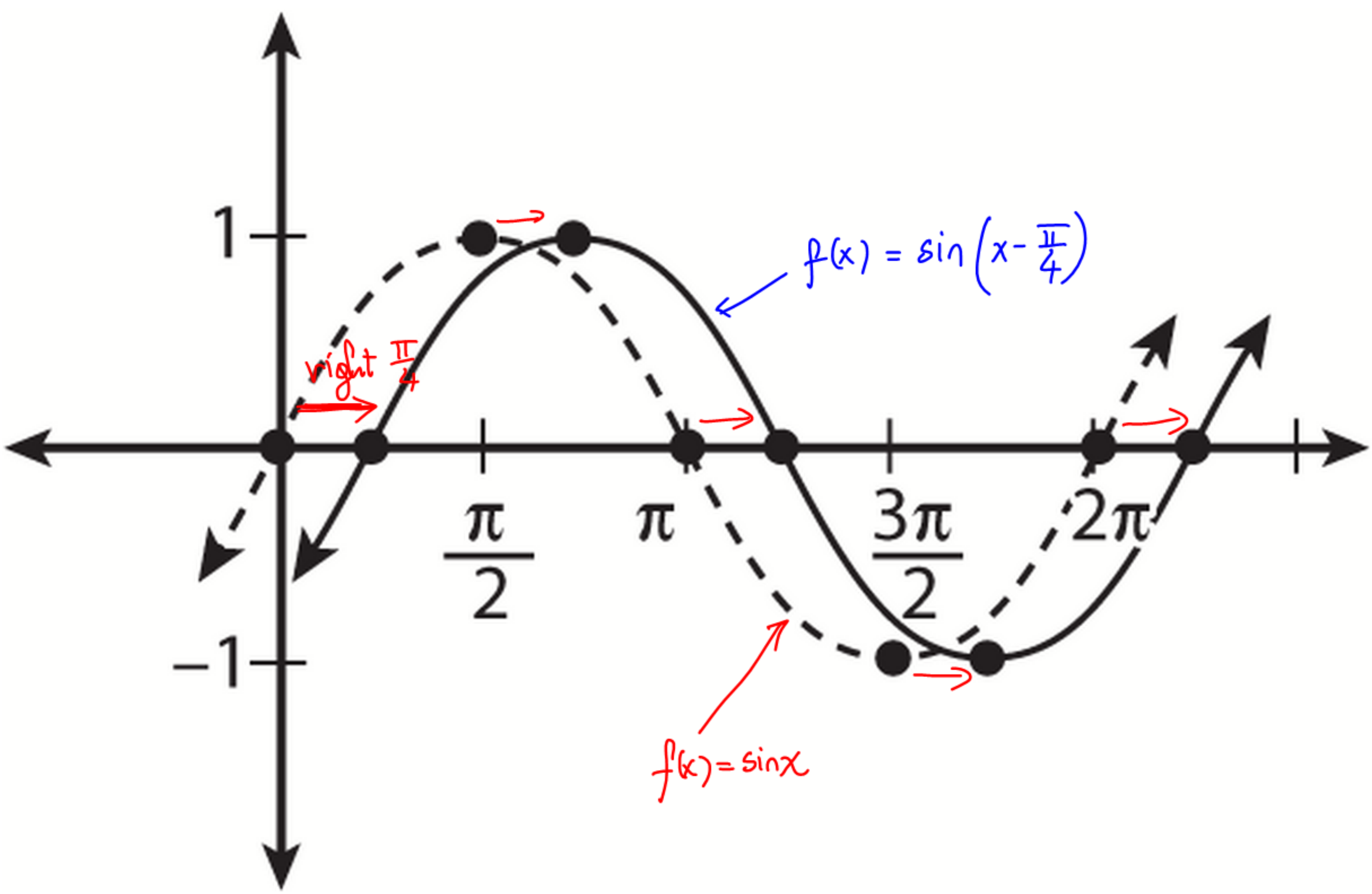
Shift the original graph D units UP if $D > 0$.
Shift the original graph D units DOWN if $D < 0$.

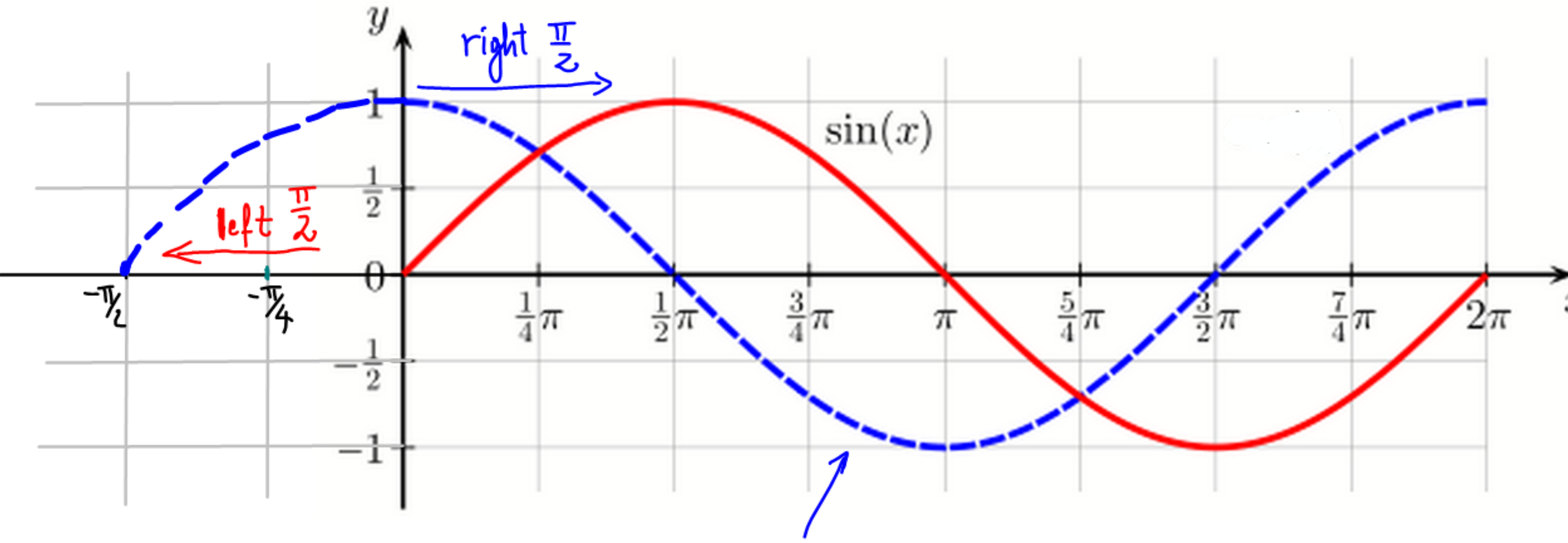


Horizontal Shift: f(x)=sin(x-C) or f(x)=cos(x-C)

Shift the original graph C units to the RIGHT if C > 0.
Shift the original graph C units to the LEFT if C < 0.






$f(x) = \sin\left(x + \frac{\pi}{2}\right) = \cos x.$

(shift $\sin x$ $\frac{\pi}{2}$ units left, you get $\cos x$)

<u>OR</u>

$f(x) = \cos\left(x - \frac{\pi}{2}\right) = \sin x$

(shift $\cos x$ $\frac{\pi}{2}$ units right, you get $\sin x$)

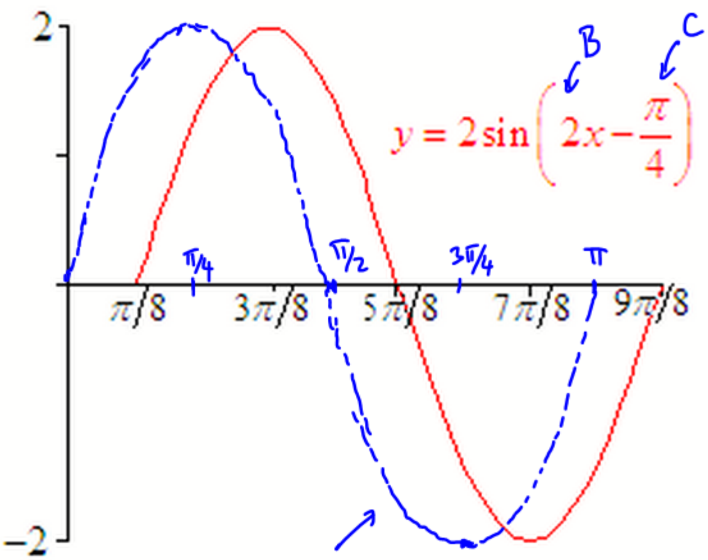
Phase shift: f(x)=sin(Bx - C)=sin( B ( x - $\frac{C}{B}$ ) )

The function will be shifted $\frac{C}{B}$ units to the right if $\frac{C}{B} > 0$.

The function will be shifted $\frac{C}{B}$ units to the left if $\frac{C}{B} < 0$.

The number $\frac{C}{B}$ is called the phase shift.

$\frac{C}{B} = \frac{\pi/4}{2} = \frac{\pi}{8} \leftarrow$ phase shift




$= 2\sin\left(2\left(x - \frac{\pi}{8}\right)\right)$

$\frac{2\pi}{2} = \pi$ is the period.

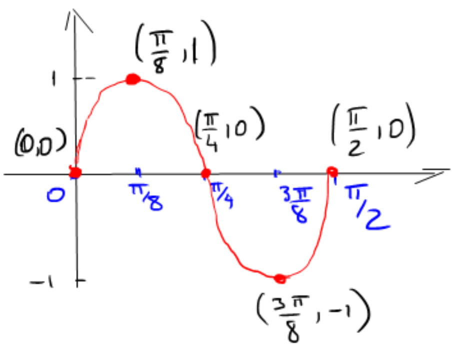
$f(x) = 2\sin(2x)$ shifted $\frac{\pi}{8}$ units to right.

To be continued on Tuesday, 03/22

**Example 3**: Write down the transformations needed to graph:

$f(x) = \sin(4x)$

Period: $\rightarrow \dfrac{2\pi}{4} = \dfrac{\pi}{2}$



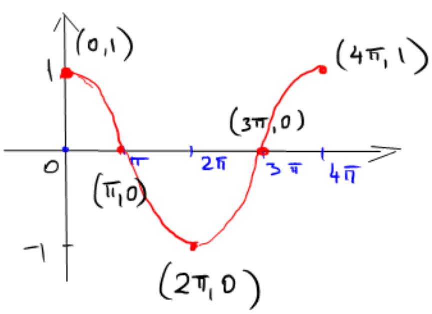

Amplitude: 1

Transformations: Horizontal shrinking by a factor of $\frac{1}{2}$

Graph: Draw the period interval over x-axis.
Divide into four equal intervals.
Plot the shape of sine function.

$f(x) = \cos\left(\frac{1}{2}x\right)$ B

Period: $\dfrac{2\pi}{1/2} = 4\pi$




Amplitude: 1

Transformations: Horizontal Stretching by a factor of 2.

$f(x) = 2\cos(x - \pi)$ A C

Vertical Stretching

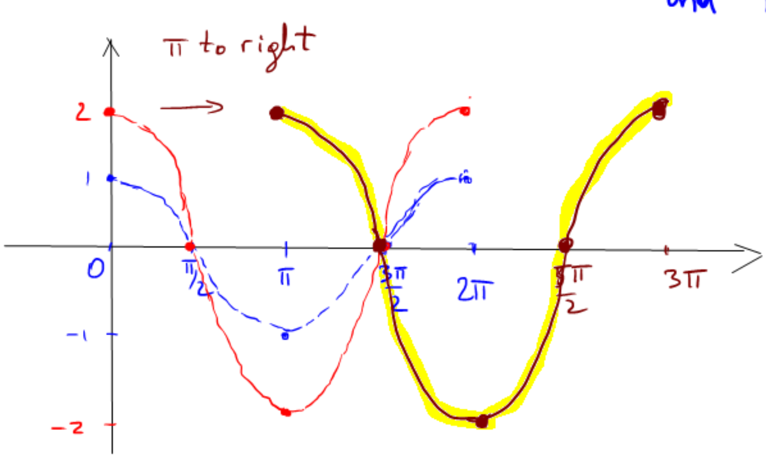
Horizontal shift

Period: $2\pi$

Amplitude: 2

$\rightarrow$ Begin with $f(x) = \cos x$.

Transformations: Vertical Stretching by a factor of 2, and horizontal shift $\pi$ units right.

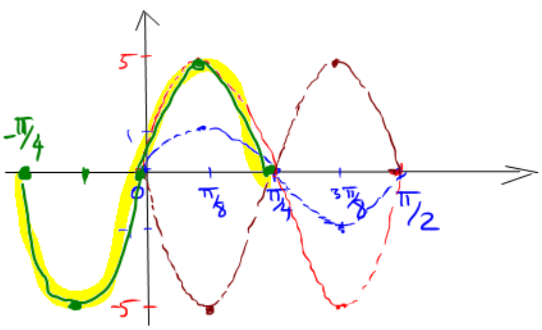


$f(x) = -5\sin(4x+\pi)$ (A = −5, B = 4, C = π)

$= -5\sin\left(4\left(x+\frac{\pi}{4}\right)\right)$

Period: $\frac{2\pi}{4} = \frac{\pi}{2}$ , Phase Shift $\frac{C}{B} = \frac{\pi}{4}$

Amplitude: 5



Transformations: · reflection wrt x-axis and vertical stretch

· Horizontal shrinking by a factor of $\frac{1}{4}$

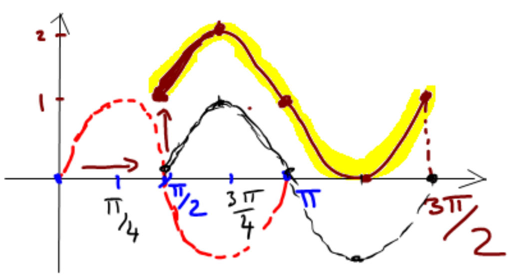
· Horizontal shift $\frac{\pi}{4}$ units to left.

$f(x) = \sin(2x-\pi)+1$ (B = 2, C = −π)

$= \sin\left(2\left(x-\frac{\pi}{2}\right)\right)+1$

Period: $\frac{2\pi}{2} = \pi$ , Phase Shift : $\frac{\pi}{2}$

Amplitude: 1



Transformations: · Horizontal Shrinking by a factor of $\frac{1}{2}$

· Horizontal shift $\frac{\pi}{2}$ units right

· Vertical shift 1 unit up.

$f(x) = 5\sin\left(\frac{\pi x}{2} - \frac{\pi}{8}\right) - 1$ (A = 5, B = π/2, C = −π/8)

$= 5\sin\left(\frac{\pi}{2}\left(x-\frac{1}{4}\right)\right)-1$

Period: $\frac{2\pi}{\pi/2} = 4$ , Phase Shift : $\frac{\pi/8}{\pi/2} = \frac{2}{8} = \frac{1}{4}$

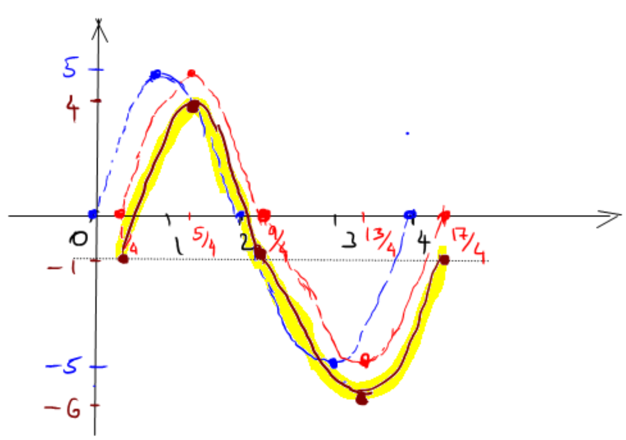
Amplitude 5

Transformations: · Vertical stretching by a factor of 5.

· Horizontal Shrinking by a factor of $2/\pi$

· Horizontal shift $\frac{1}{4}$ unit right.

· Vertical shift 1 unit down



# Five Basic Points: How to find them?

It can be helpful to identify the starting and ending points for one period of the graph of a function that has a phase shift. To do this, solve the equations $Bx - C = 0$ and $Bx - C = 2\pi$.

$f(x) = 5\cos(2x - \pi)$ ;  starting point: $2x - \pi = 0 \rightarrow x = \frac{\pi}{2}$ → phase shift

→ find Period  ending point: $2x - \pi = 2\pi \rightarrow 2x = 3\pi \rightarrow x = 3\pi/2$ → the end

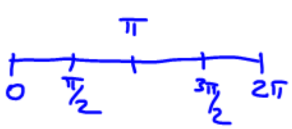
→ divide into 4 equal intervals, then apply transformations!!!

You will need to identify the transformations required to change a basic sine or cosine function to the desired one. You must know the **five key points** on a basic sine function and the **five key points** on a basic cosine function. Using the information about the amplitude, reflections, vertical and horizontal stretching or shrinking and vertical and horizontal translations, you will be able to correctly plot the translated key points and sketch the desired function.

**Example 4:** Sketch over one period: $f(x) = 4\sin(x)$

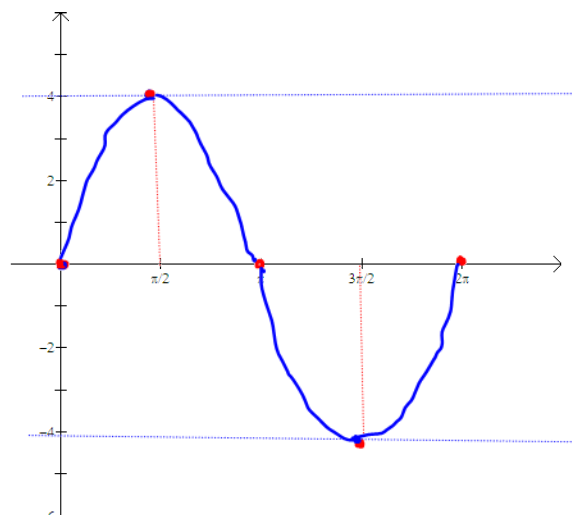
→ period $= 2\pi$

→ divide into 4 equal intervals



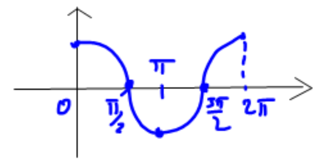

→ amplitude $= 4$

→ vertical stretch



Steps to graph $f(x) = 5\cos(2x - \pi)$:

→ We are working with $y = \cos x$




Basic Shape.

→ Period $\frac{2\pi}{2} = \pi$


← New Five Basic Points

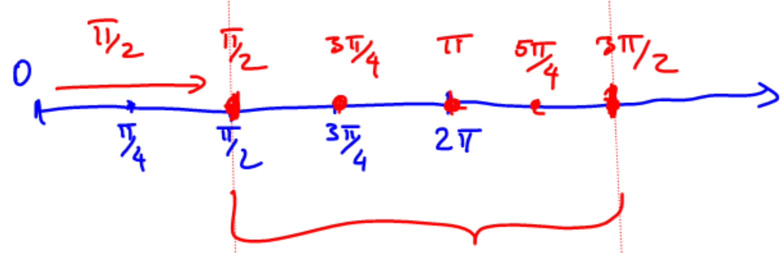
→ Phase Shift $\frac{C}{B} = \frac{\pi}{2}$

$2x - \pi = 0$

$x = \frac{\pi}{2}$

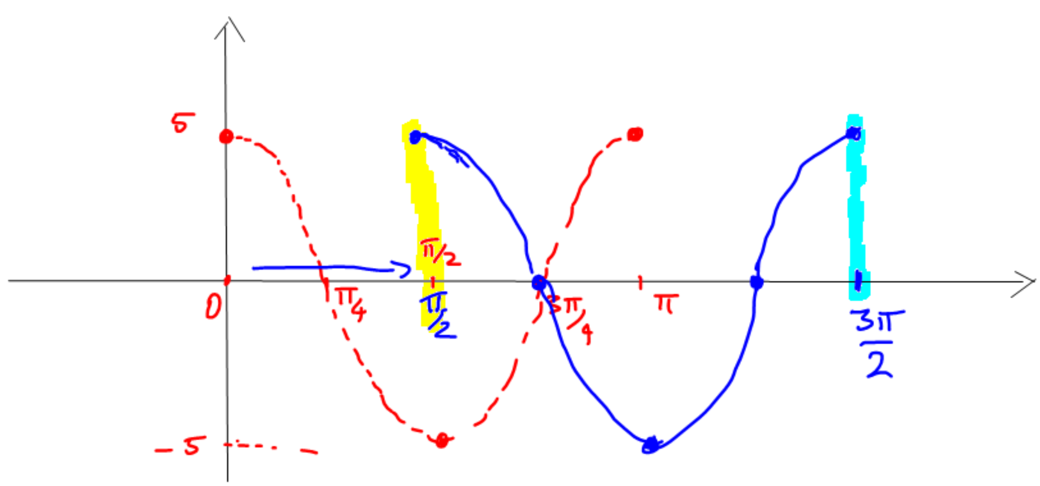
$2x - \pi = 2\pi$

$x = \frac{3\pi}{2}$




Shifted $\frac{\pi}{2}$ units Right.

→ Put on the graph:



**Example 5:** Sketch over one period: $f(x) = -\cos\left(\frac{x}{2}\right)$

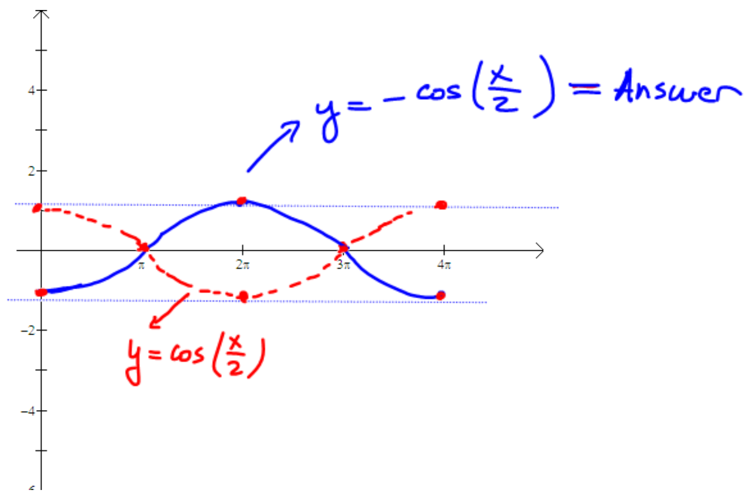
$\rightarrow$ period $= \frac{2\pi}{1/2} = 4\pi$

$0 \quad \pi \quad 2\pi \quad 3\pi \quad 4\pi$

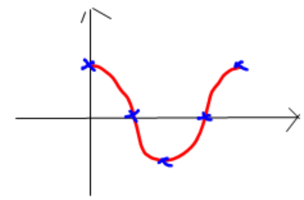
horizontal stretching

$\rightarrow$ amplitude $= 1$

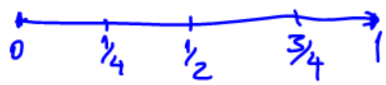
$\rightarrow$ reflect wrt. x-axis



**Example 6:** Sketch over one period: $f(x) = \underset{A}{4}\cos(\underbrace{2\pi}_{B} x) \underset{D}{-1}$



→ period $= \frac{2\pi}{2\pi} = 1$

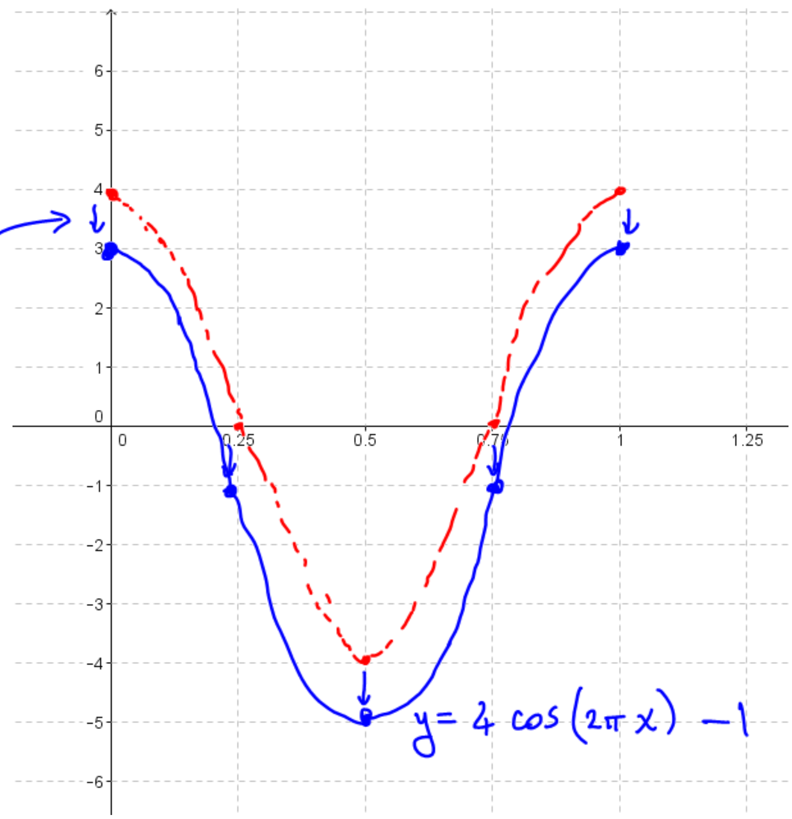



→ amplitude = 4

Vertical stretching of cosine

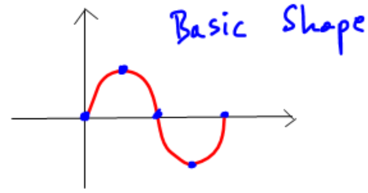
→ at the end,

shift the function 1 down.

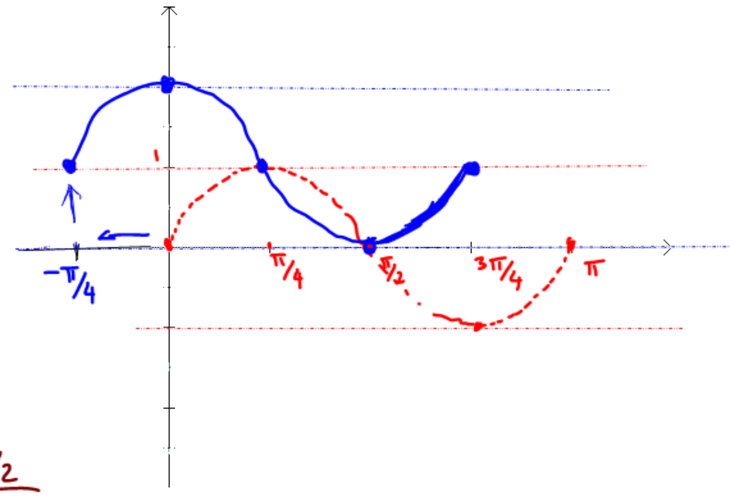


**Example 7:** Sketch over one period: $f(x)=\sin\left(2x+\frac{\pi}{2}\right)+1$

A B C D




- $A = 1 \Rightarrow$ Normal Stretch



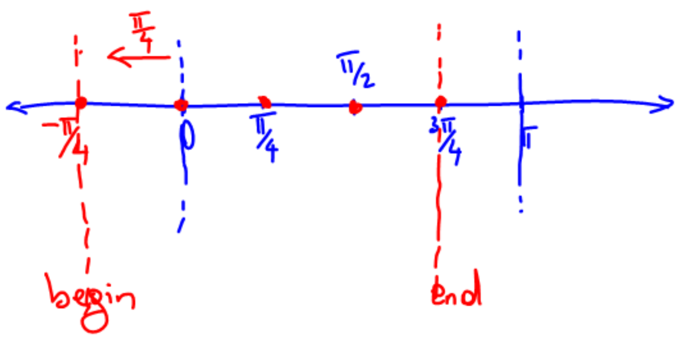

- $B = 2 \Rightarrow$ period $= \frac{2\pi}{2} = \pi$

Horizontal Shrinking

- $C = -\frac{\pi}{2} \Rightarrow$ phase shift $= \frac{C}{B} = \frac{-\pi/2}{2}$

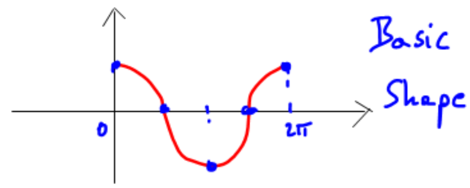
$= -\frac{\pi}{4}$

Horizontal Shift $\frac{\pi}{4}$ units to left.

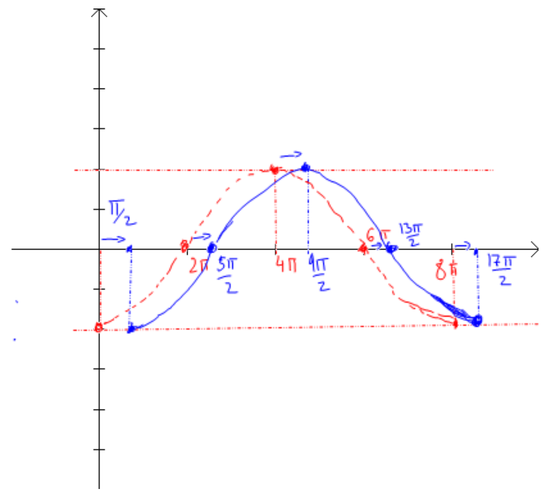



- $D = 1 \Rightarrow$ Vertical Shift 1 unit up.

**Example 8:** Sketch over one period: $f(x) = \overset{A}{-2}\cos\left(\overset{B}{\frac{1}{4}}x - \overset{C}{\frac{\pi}{8}}\right)$




• $A = -2 \Rightarrow$ Vertical Stretching and reflection wrt x-axis.

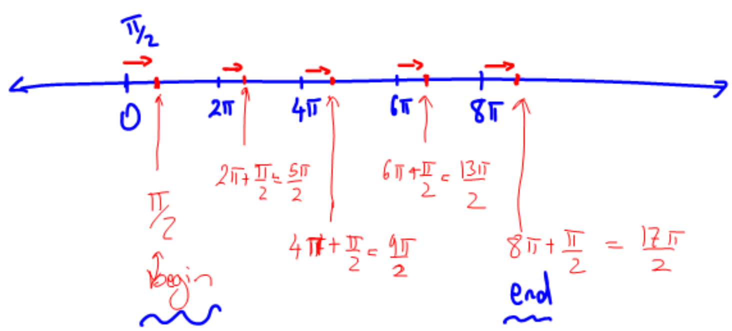



• $B = \frac{1}{4} \Rightarrow$ period $= \frac{2\pi}{1/4} = 8\pi$

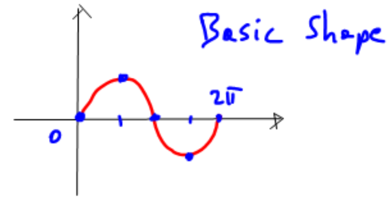
Horizontal Stretching

• $C = \frac{\pi}{8} \Rightarrow$ phase shift $= \frac{C}{B} = \frac{\pi/8}{1/4} = \frac{\pi}{2}$

Horizontal shift $\frac{\pi}{2}$ units right.



**Exercise:** Sketch over one period: $f(x) = \underset{A}{2}\sin\left(\underset{B}{\frac{\pi x}{2}} - \underset{C}{\frac{\pi}{8}}\right)\underset{D}{-1}$




- $A=2 \Rightarrow$ Vertical Stretching.

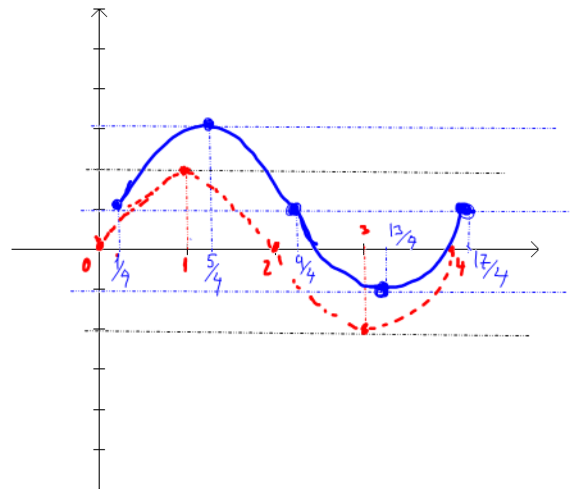
- $B=\frac{\pi}{2} \Rightarrow$ period $= \frac{2\pi}{\pi/2} = \boxed{4}$

  Horizontal Shrinking.

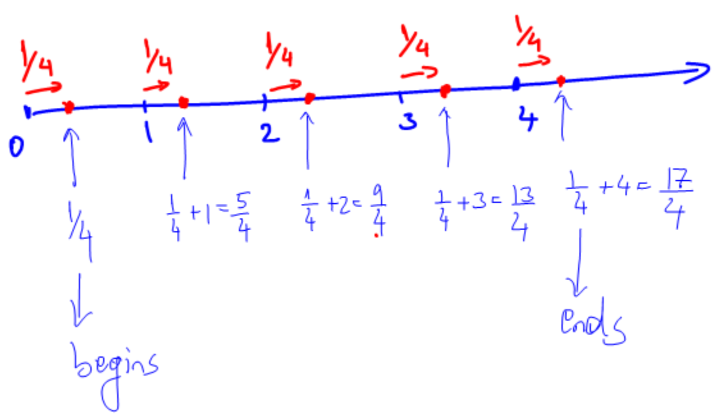
- $C=\frac{\pi}{8} \Rightarrow$ phase Shift $= \frac{\pi/8}{\pi/2}$

  $= \frac{\cancel{\pi}}{8} \cdot \frac{2}{\cancel{\pi}} = \boxed{\frac{1}{4}} \Rightarrow$ Horizontal Shift $\frac{1}{4}$ unit to right.

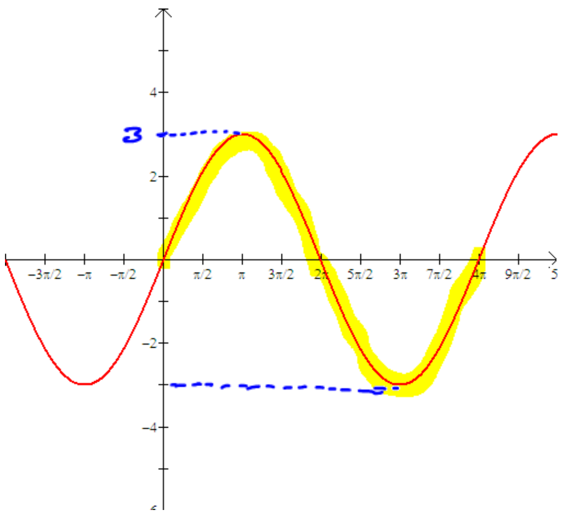
- $D=-1 \Rightarrow$ Vertical Shift 1 unit Down.




Period $= 4$



**Example 9:** Consider the graph: Write an equation of the form $f(x) = A\sin(Bx - C) + D$ and an equation of the form $f(x) = A\cos(Bx - C) + D$ which could be used to represent the graph. Note: these answers are not unique!




It is a sine function:

→ amplitude = 3 ⇒ $A = 3$

→ period = $4\pi$ ⇒ $B = \frac{1}{2}$

$$\frac{2\pi}{B} = \frac{4\pi}{1}$$

$$2\pi = B \cdot 4\pi$$

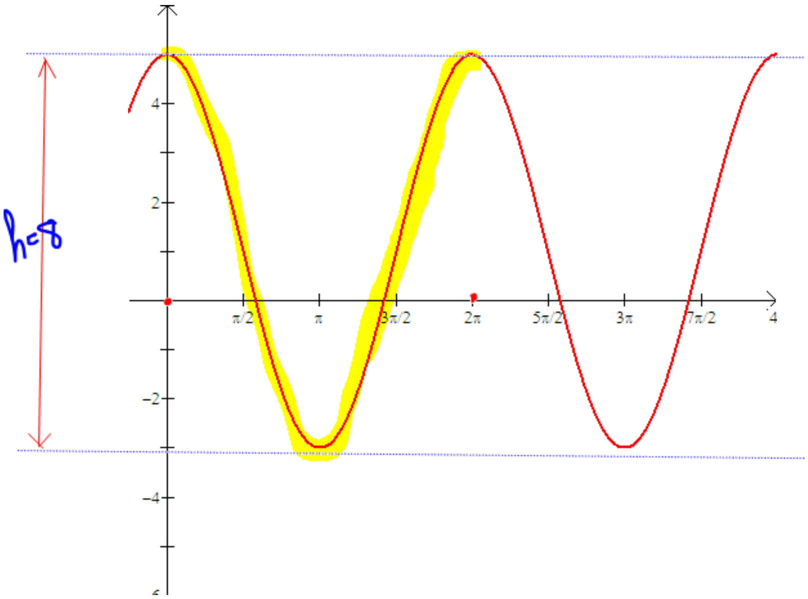
$$\Rightarrow B = \frac{2\pi}{4\pi} = \frac{1}{2}$$

→ There is no horizontal shift or vertical shift. $C = D = 0$

$$\boxed{\text{Answer} \Rightarrow y = 3\sin\left(\tfrac{1}{2}x\right)}$$

**Example 10:** Consider the graph: Write an equation of the form $f(x) = A\sin(Bx - C) + D$ and an equation of the form $f(x) = A\cos(Bx - C) + D$ which could be used to represent the graph. Note: these answers are not unique!

Max = 5, min = −3




$\Rightarrow \boxed{f(x) = 4\cos(x) + 1}$

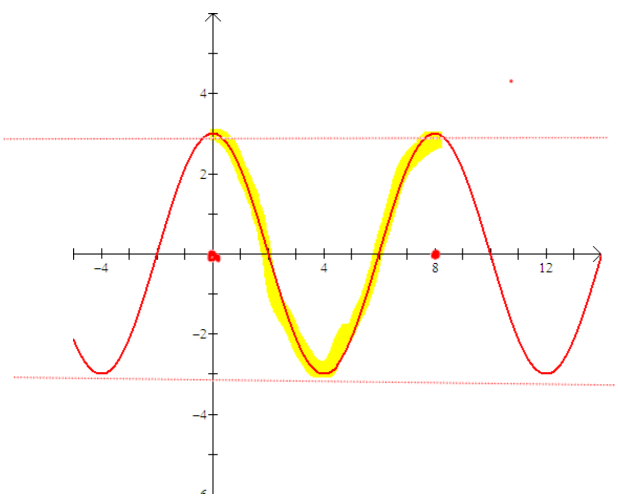
- It looks like a cosine function shifted 1 unit up $\Rightarrow D = 1$.

  $D = \frac{(-3)+5}{2} = 1$

- The distance between minimum and maximum is 8, thus amplitude $A = \frac{8}{2} = 4$.

- The length of interval where the shaded cosine shape gives period $= 2\pi$. $\Rightarrow B = 1$

**Exercise:** Consider the graph: Write an equation of the form $f(x) = A\sin(Bx - C) + D$ and an equation of the form $f(x) = A\cos(Bx - C) + D$ which could be used to represent the graph. Note: these answers are not unique!




- It looks like cosine function.

- The amplitude is $A = 3$

- The period is $\frac{2\pi}{B} = 8 \Rightarrow B = \frac{\pi}{4}$

- Hence, $\boxed{f(x) = 3\cos\left(\frac{\pi}{4}x\right)}$