should be 1-1 → Restrictions of domains

Section 5.4 - Graphs of Inverse Trigonometric Functions

Example :

Recall:

$\sin(\sin^{-1}(x)) = x$ when $x \in [-1, 1]$

$\cos(\cos^{-1}(x)) = x$ when $x \in [-1, 1]$

$\tan(\tan^{-1}(x)) = x$ when $x \in (-\infty, \infty)$

• $\sin^{-1}(x) = \theta$ is an angle in QI or QIV.

$\sin^{-1}(\sin(x)) = x$ when $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}(\cos(x)) = x$ when $x \in [0, \pi]$

$\tan^{-1}(\tan(x)) = x$ when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

• $\sin^{-1}\left(\underbrace{\sin\left(\overset{QI}{\frac{\pi}{6}}\right)}_{1/2}\right) = \overset{QI}{\frac{\pi}{6}}$

BUT

• $\sin^{-1}\left(\underbrace{\sin\left(\overset{QII}{\frac{5\pi}{6}}\right)}_{1/2}\right) = \overset{QI}{\frac{\pi}{6}}$

ALSO

• $\sin^{-1}\left(\underbrace{\sin\left(\overset{QIV}{\frac{11\pi}{6}}\right)}_{-\frac{1}{2}}\right) = \overset{QIV}{-\frac{\pi}{6}}$

Example 1: $\tan^{-1}\left(\underbrace{\tan\left(\overset{QII}{\frac{2\pi}{3}}\right)}_{-\sqrt{3}}\right) = \tan^{-1}(-\sqrt{3}) = \overset{QIV}{\boxed{-\frac{\pi}{3}}}$

Example 2: $\arcsin\left(\underbrace{\sin\left(\overset{QIV}{\frac{5\pi}{3}}\right)}_{-\frac{\sqrt{3}}{2}}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \overset{QIV}{\boxed{-\frac{\pi}{3}}}$

Example 3: $\tan\left(\underbrace{\cot^{-1}\left(\frac{2}{5}\right)}_{\theta}\right) = \tan\theta = \boxed{\frac{5}{2}}$

$\cot^{-1}\left(\frac{2}{5}\right) = \theta \iff \cot\theta = \frac{2}{5}$

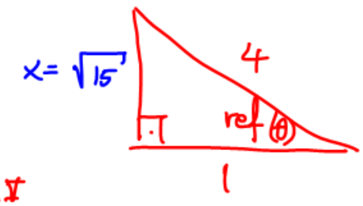
$\implies \tan\theta = \frac{5}{2}$

Example 4: $\sin\left(\underbrace{\cos^{-1}\left(-\frac{1}{4}\right)}_{\theta}\right) = \sin\theta = \boxed{\frac{\sqrt{15}}{4}}$

QII $\longrightarrow$ positive

$\cos^{-1}\left(\frac{-1}{4}\right) = \theta$ $\Rightarrow \sin\theta = \frac{\sqrt{15}}{4}$

$\Rightarrow \cos\theta = \frac{-1}{4}$



$x^2 + 1^2 = 4^2$

$x = \sqrt{15}$

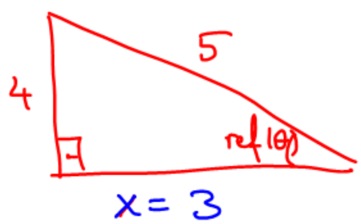
QIV

Example 5: $\tan\left(\underbrace{\sin^{-1}\left(-\frac{4}{5}\right)}_{\theta}\right) = \tan\theta = \boxed{-\frac{4}{3}}$

(negative)

$\sin^{-1}\left(-\frac{4}{5}\right) = \theta$

$\sin\theta = -\frac{4}{5}$



$x^2 + 4^2 = 5^2$

$x = 3$

$\Rightarrow \tan\theta = \frac{4}{3}$ (ref)

QI

Example 6: $\tan\left(\underbrace{\sec^{-1}(2)}_{\theta}\right) = \tan\theta = \boxed{\tan\left(\frac{\pi}{3}\right) = \sqrt{3}}$

(positive)

$\sec^{-1}(2) = \theta$

$\sec\theta = 2$

$\frac{1}{\cos\theta} = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

Example 7: $\sin\left(\underbrace{\cos^{-1}(-4)}\right) =$ undefined.

???

undefined, should be a number from $[-1, 1]$

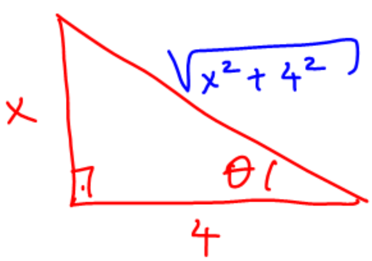
Example 8: Simplify $\cos\left(\underbrace{\arctan\left(\frac{1}{4}x\right)}_{\theta}\right)$ where $x > 0$.

QI

$\Rightarrow \cos\theta = \boxed{\dfrac{4\sqrt{x^2+16}}{x^2+16}}$

$\tan^{-1}\left(\frac{x}{4}\right) = \theta$

$\Leftrightarrow \tan\theta = \frac{x}{4}$

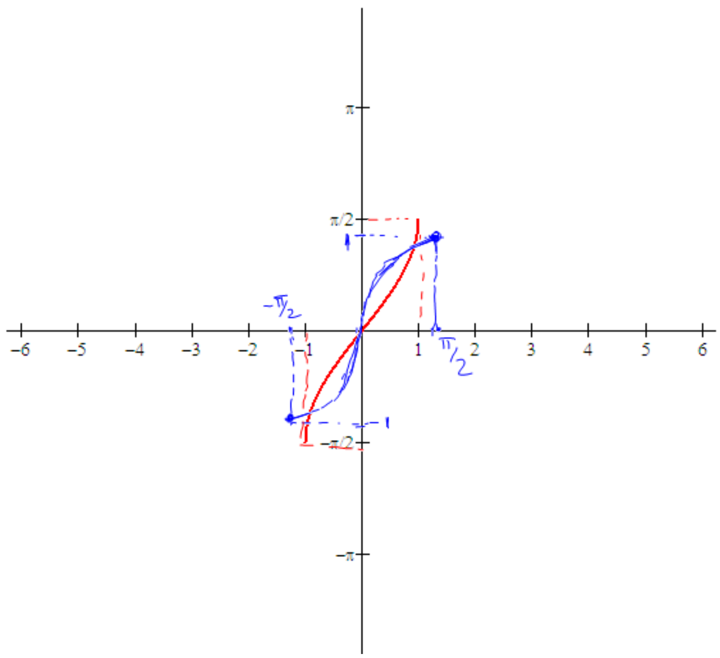


$\Rightarrow \cos\theta = \dfrac{4}{\sqrt{x^2+16}} = \dfrac{4\sqrt{x^2+16}}{x^2+16}$

Graphs of Inverse Trigonometric Functions

Earlier, we looked briefly at the graphs of the inverse sine, inverse cosine and inverse tangent functions. Here's a recap.

We note the inverse sine function as $f(x) = \sin^{-1}(x)$ or $f(x) = \arcsin(x)$. The domain of the inverse function will be [-1, 1] and the range of the inverse function will be $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Here is the graph of $f(x) = \sin^{-1}(x)$:



$f(x) = \sin x$

1-1

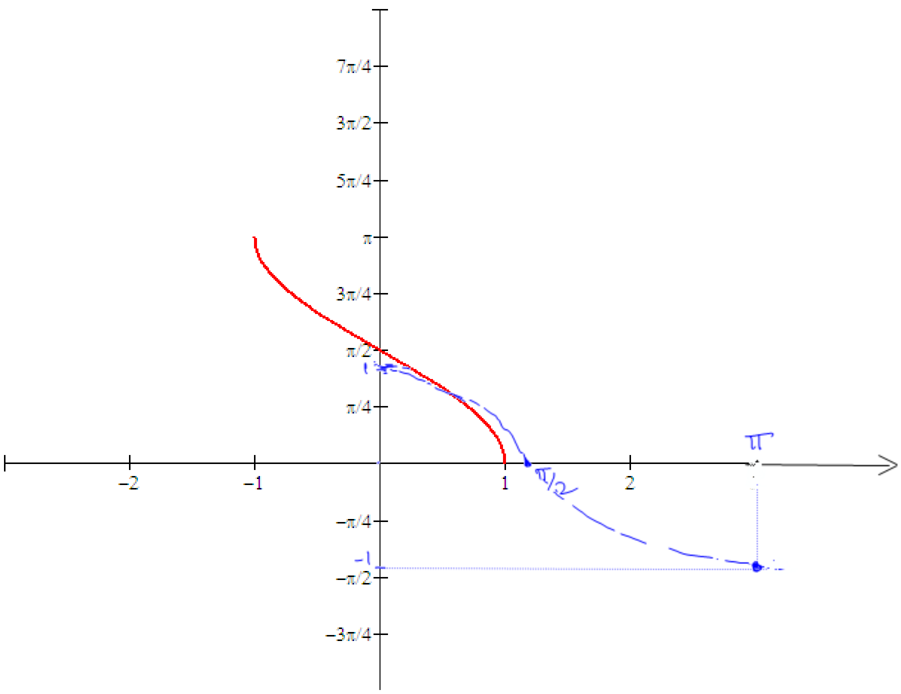
$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1,1]$

$f^{-1}(x) = \sin^{-1}(x)$

$[-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

range in QI or QIV

We note the inverse cosine function as $f(x) = \cos^{-1}(x)$ or $f(x) = \arccos(x)$. The domain of the inverse function will be [-1, 1] and the range of the inverse function will be $[0, \pi]$. Here is the graph of $f(x) = \cos^{-1}(x)$:



$f(x) = \cos x$

1-1

$[0, \pi] \longrightarrow [-1, 1]$

$f^{-1}(x) = \cos^{-1}(x)$

$[-1, 1] \longrightarrow [0, \pi]$

range in QI or QII.

We note the function as $f(x) = \tan^{-1}(x)$ or $f(x) = \arctan(x)$. The domain of the inverse function is $(-\infty, \infty)$ and the range is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$. Here is the graph of the inverse tangent function:

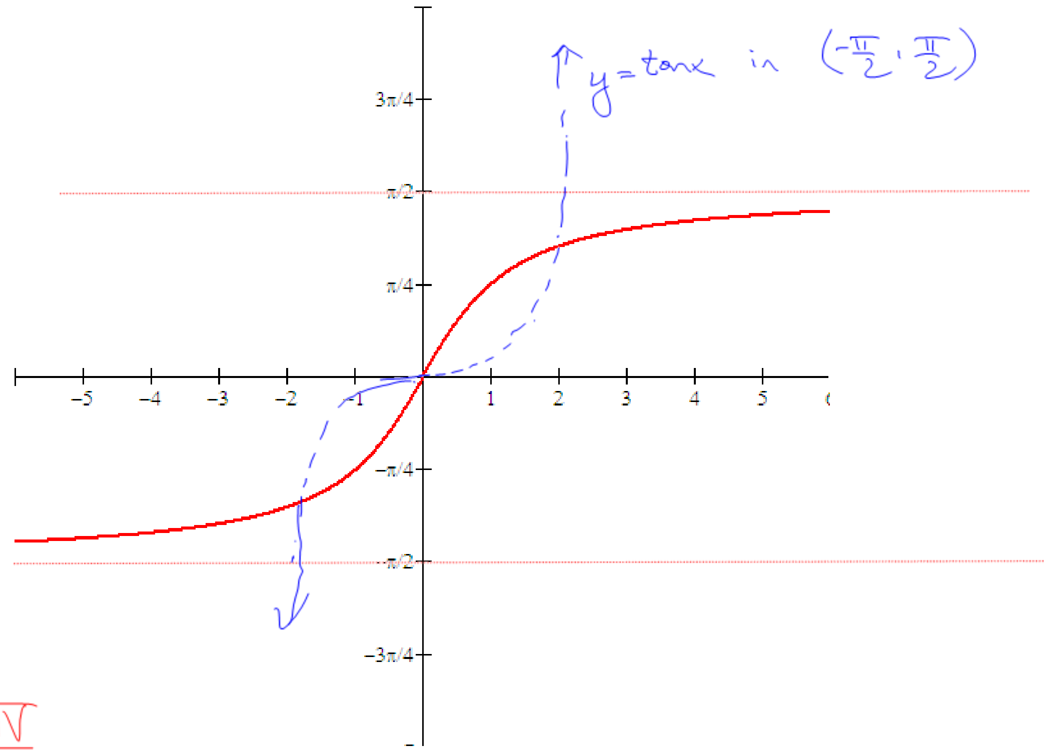
$f(x) = \tan x$

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{1-1} (-\infty, \infty)$

$f^{-1}(x) = \tan^{-1}(x)$

$(-\infty, \infty) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

range in QI or QIV

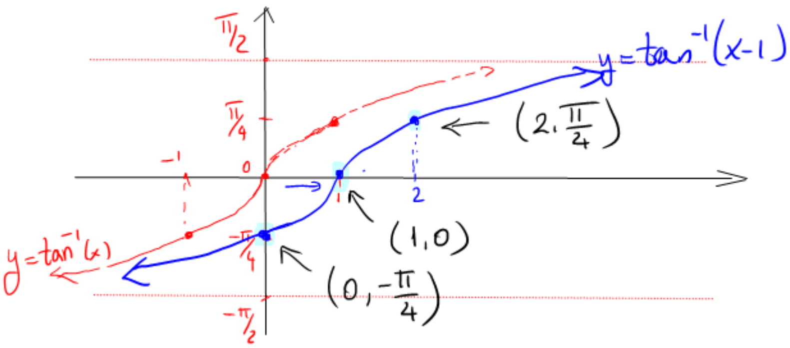


$\boxed{y = \pm\frac{\pi}{2} \text{ are horizontal Asymptotes}}$

You can use graphing techniques learned in earlier lessons to graph transformations of the basic inverse trig functions.

Example 1: Which of the following points is on the graph of $f(x) = \arctan(x-1)$? $= \underbrace{\tan^{-1}(x-1)}_{\text{shift 1 unit right}}$

A) $\left(\frac{\pi}{4}, 0\right)$ B) $\left(0, \frac{\pi}{4}\right)$ C) $\left(0, -\frac{\pi}{4}\right)$ D) $\left(2, \frac{\pi}{4}\right)$

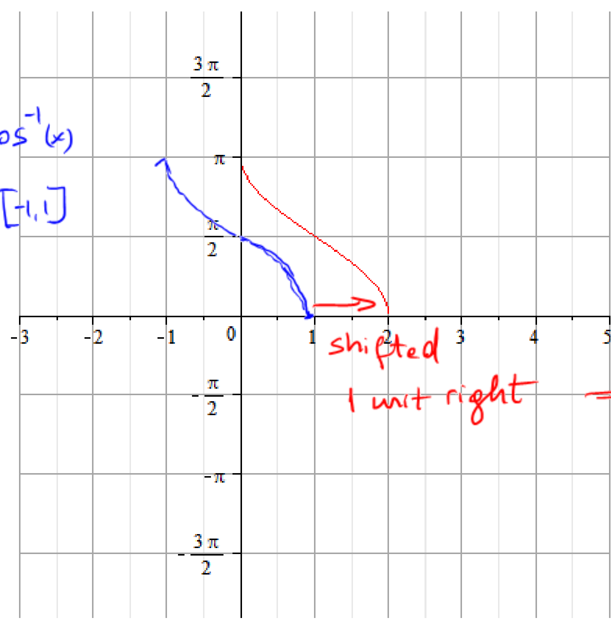


Graphing simplifies your work!

Example 2: Which of the following can be the function whose graph is given below?

$f(x) = \cos^{-1}(x)$

$[0, \pi] \to [-1, 1]$



A) $f(x) = \cos^{-1}(x-1)$

B) $f(x) = \sin^{-1}(x-1)$

C) $f(x) = \cos^{-1}(x+1)$

D) $f(x) = \sin^{-1}(x+1)$

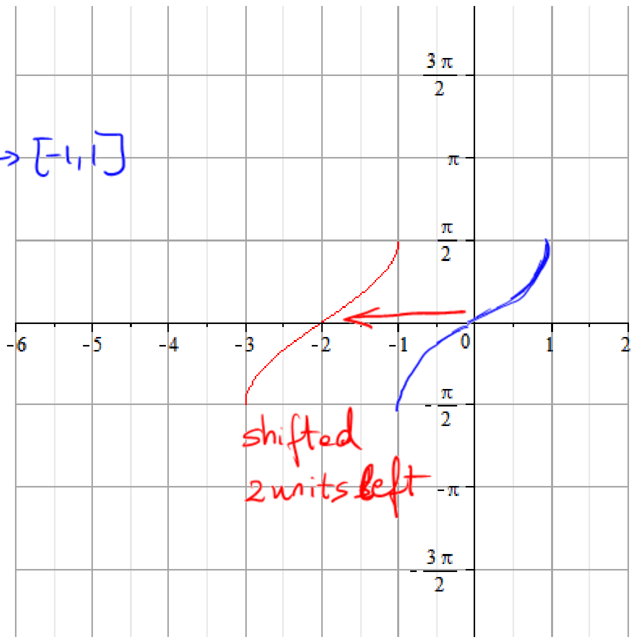
E) $f(x) = \tan^{-1}(x-1)$

$\Longrightarrow f(x) = \cos^{-1}(x-1)$

Example 3: Which of the following can be the function whose graph is given below?

$f(x) = \sin^{-1} x$

$[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$



A) $f(x) = \cos^{-1}(x-2)$

B) $f(x) = \sin^{-1}(x-2)$

C) $f(x) = \cos^{-1}(x+2)$

D) $f(x) = \sin^{-1}(x+2)$

E) $f(x) = \tan^{-1}(x+2)$

$\Rightarrow f(x) = \sin^{-1}(x+2)$

Modeling Using Sinusoidal Functions

Sine and cosine functions model many real-world situations. Physical phenomenon such as tides, temperatures and amount of sunlight are all things that repeat themselves, and so are easily modeled by sine and cosine functions (collectively, they are called "sinusoidal functions").

Here are some other situations that can be modeled by a sinusoidal function:

- Suppose you are on a Ferris wheel at a carnival. Your height (as you are sitting in your seat) varies sinusoidally.
- Suppose you are pushing your child as s/he sits in a swing. Your child's height varies sinusoidally.
- The motion of a swinging pendulum varies sinusoidally.
- Stock prices sometimes vary sinusoidally.

We'll work a couple of examples involving sinusoidal variation.

We know all of them.

Recall: Given the functions $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$;

The period is: $\frac{2\pi}{B}$

The amplitude is: $|A|$

The vertical shift is: D (up if positive)

The horizontal shift is $\frac{C}{B}$ (to the right if positive)

We'll learn how to connect to real-world situations!

Example 1: Determine the equation of the ~~sine~~ function which has amplitude is 5, the phase shift is 4 to the left, the vertical shift is 3 down, and the period is 2.

$f(x)=5\sin(\pi(x+4))-3$

$$f(x) = A\sin(Bx - C) + D$$

$$= A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D$$

$\boxed{D = -3}$

$\boxed{A = 5}$

period $= \frac{2\pi}{B} = 2$

$\Rightarrow \boxed{B = \pi}$

phase shift $= \frac{C}{B} = -4$

$\Rightarrow \boxed{C = -4\pi}$

$$\Rightarrow \boxed{f(x) = 5\sin(\pi x + 4\pi) - 3}$$

Application

Example 2: The number of hours of daylight in Boston is given by

$f(x) = 3\sin\left(\frac{2\pi}{365}(x-79)\right)+12$ where x is the number of days after January 1. What is the:

a. amplitude? $\boxed{A = 3}$

b. period? $= \frac{2\pi}{B} = \frac{2\pi}{2\pi/365} = \boxed{365 = \text{a year}}$

c. maximum value of f(x)?

max. value occurs when sine have value $= 1$

$\Rightarrow 3 \cdot 1 + 12 = \boxed{15 \text{ hours} \leftrightarrow \text{longest day of the year.}}$

Example 3: The function $P(t) = 120 + 40\sin(2\pi\ t)$ models the blood pressure (in millimeters of mercury) for a person who has a blood pressure of 160/90 (which is high); t represents seconds. What is the period of this function? What is the amplitude?

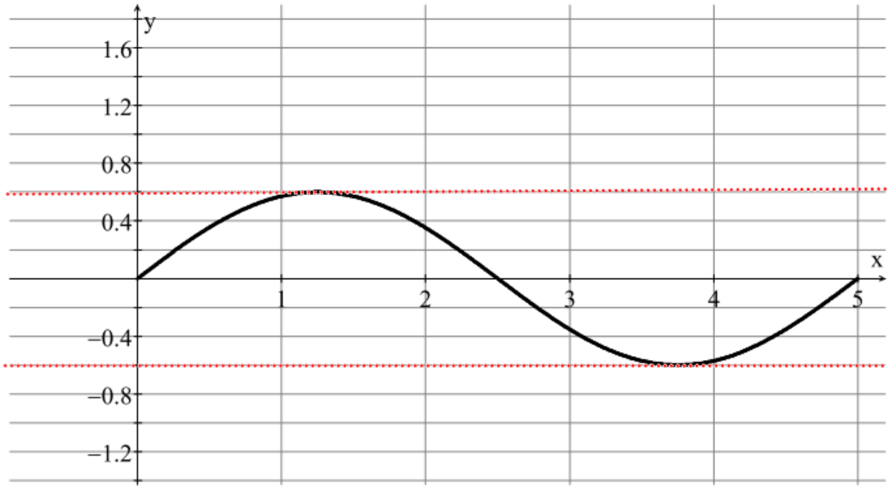
exercise

$P(t) = \underset{A}{40} \sin(\underset{B}{2\pi} t) + \underset{D}{120}$

$\Rightarrow$ period $= \frac{2\pi}{B} = \frac{2\pi}{2\pi} = 1$

amplitude $= 40$.

Example 4: Determine the function of the form $f(x) = A\sin(Bx)$ given the following graph:



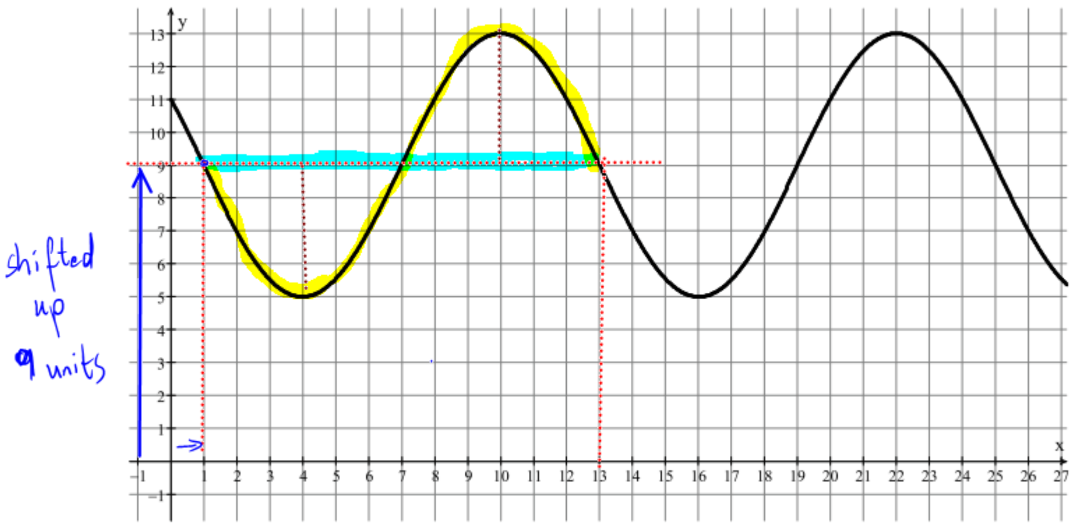
From graph, it is sine with no horizontal/vertical shift.
$C=0$ $D=0$

$\Rightarrow$ period $= 5 = \frac{2\pi}{B}$ $\Rightarrow$ $\boxed{B = \frac{2\pi}{5} = 0.4\pi}$

$\Rightarrow$ amplitude $= A = 0.6$

$\Rightarrow$ $\boxed{f(x) = 0.6 \sin(0.4\pi x)}$

Example 5: Determine the function of the form $f(x) = A\sin(Bx - C) + D$ given the following graph:



• Look for sine graph.

• Vertical shift $\boxed{D = 9}$

• amplitude = 4

reflected $\Rightarrow \boxed{A = -4}$

• period = 13 − 1 = 12

$12 = \frac{2\pi}{B} \Rightarrow \boxed{B = \frac{\pi}{6}}$

• phase shift = 1 right

$\frac{C}{B} = 1 \Rightarrow \boxed{C = B = \frac{\pi}{6}}$

$$\Rightarrow \boxed{y = -4\sin\left(\frac{\pi}{6}x - \frac{\pi}{6}\right) + 9}$$

Example 6: Assume that you are aboard a research submarine doing submerged training exercises in the Pacific Ocean. At time t = 0 you start porpoising (alternately deeper and then shallower). At time t = 4 min you are at your deepest, y = – 1000 m. At time t = 9 min you next reach your shallowest, y = –200 m. Assume that y varies sinusoidally with time. Find an equation expressing y as a function of t.

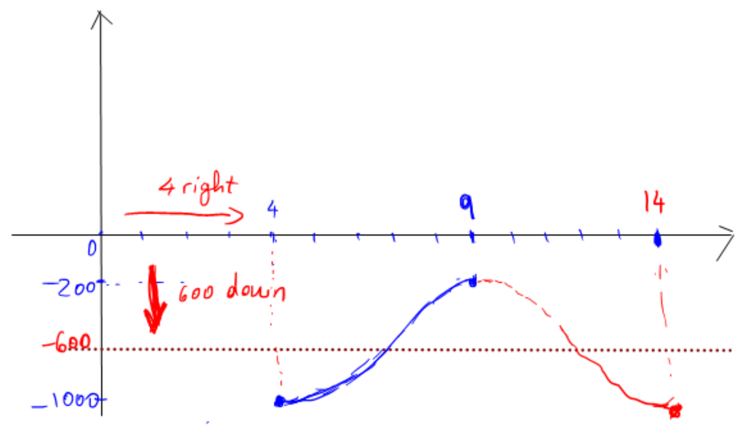
A) $f(t) = -600\cos\left(\frac{\pi}{5}(t-4)\right) - 200$

B) $f(t) = -400\cos\left(\frac{\pi}{5}(t-4)\right) - 600$

C) $f(t) = -200\cos\left(\frac{\pi}{5}(t-4)\right) - 400$

D) $f(t) = -600\cos\left(\frac{\pi}{5}(t-4)\right) - 400$

E) A) $f(t) = -400\cos\left(\frac{\pi}{15}(t-9)\right) - 600$



- From graph; amplitude $= \frac{1000-200}{2} = 400$

  Since reflected $\Rightarrow \boxed{A = -400}$

- period $= 14 - 4 = 10$, $\frac{2\pi}{B} = 10 \Rightarrow \boxed{B = \frac{\pi}{5}}$

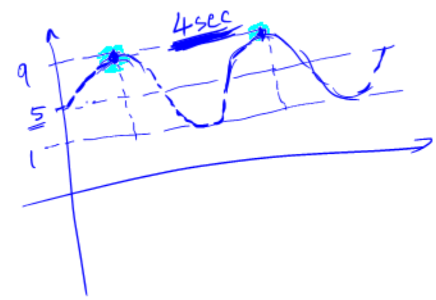
- shifted 4 units right, $\frac{C}{B} = 4 \Rightarrow \boxed{C = \frac{4\pi}{5}}$

- shifted down 600 units $\Rightarrow \boxed{D = -600}$

Note: The vertical shift is always the midpoint between the minimum and maximum value of the trigonometric fn. $D = \frac{(-200)+(-1000)}{2} = -600$

(Extra) Example: A signal buoy in the Gulf of Mexico bobs up and down with the height h of its transmitter (in feet) above sea level modeled by $h(t) = A\sin(Bt) + 5$. During a small squall its height varies from 1 ft to 9 ft and there are 4 seconds from one 9-ft height to the next. What are the values of the constants A and B?

$h(t) = A \sin(Bt) + 5$



- amplitude $= \frac{9-1}{2} = 4 \Rightarrow \boxed{A = 4}$
- period $= 4 = \frac{2\pi}{B} \Rightarrow \boxed{B = \frac{\pi}{2}}$