Warm-Up :

We'll learn $\sin(A+B)$ or $\sin(A-B)$

$\cos(A+B)$ or $\cos(A-B)$

$\tan(A+B)$ or $\tan(A-B)$, and so on...

Never forget:

$$\sin(A+B) \neq \sin(A) + \sin(B)$$

ex.

$\sin(30^\circ + 60^\circ)$

$= \sin(90^\circ)$

$= 1$

$\sin(30^\circ) + \sin(60^\circ)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}$

$= \frac{1+\sqrt{3}}{2}$

$$\Longrightarrow \quad \sin(30^\circ + 60^\circ) \neq \sin(30^\circ) + \sin(60^\circ)$$

You can find similar examples to show that trigonometric functions are <u>not</u> linear.

Section 6.1 - Sum and Difference Formulas

Note: $\sin(A+B) \neq \sin(A)+\sin(B)$
$\cos(A+B) \neq \cos(A)+\cos(B)$

Sum and Difference Formulas for Sine, Cosine and Tangent

Very important

$$\boxed{\begin{aligned}
&\sin(A+B) = \sin A\cos B + \sin B\cos A \\
&\sin(A-B) = \sin A\cos B - \sin B\cos A \\
&\cos(A+B) = \cos A\cos B - \sin A\sin B \\
&\cos(A-B) = \cos A\cos B + \sin A\sin B \\
&\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B} \\
&\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}
\end{aligned}}$$

Why do we need them?
Well, we can calculate a lot of angles without using calculator.

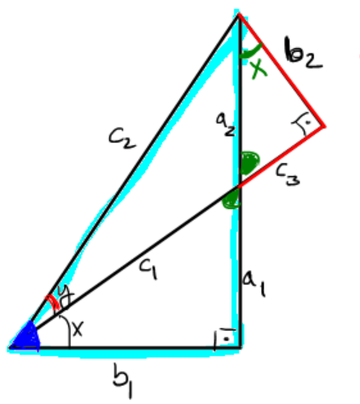
$$\sin(75^\circ) = \sin(30^\circ+45^\circ) = \sin 30^\circ\cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4} \Leftarrow \text{exact value.}$$

How to prove one of the identities, say:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$



$$\boxed{\sin(x+y) = \frac{a_1 + a_2}{c_2}}$$

$$\sin x = \frac{a_1}{c_1} = \frac{c_3}{a_2} \qquad \sin y = \frac{b_2}{c_2}$$

$$\cos x = \frac{b_1}{c_1} = \frac{b_2}{a_2} \qquad \cos y = \frac{c_1 + c_3}{c_2}$$

Cross-product

$c_1 \cdot c_3 = a_1 a_2$

$$\sin x \cos y + \cos x \sin y = \frac{c_3}{a_2} \cdot \frac{(c_1+c_3)}{c_2} + \frac{b_2}{a_2} \cdot \frac{b_2}{c_2}$$

$$= \frac{\overset{a_1 a_2}{\overbrace{c_3 \cdot c_1}} + \boxed{c_3^2 + b_2^2}}{a_2 \cdot c_2} \qquad c_3^2 + b_2^2 = a_2^2$$

$$= \frac{a_1 a_2 + a_2^2}{a_2 \cdot c_2} = \frac{\cancel{a_2}(a_1 + a_2)}{\cancel{a_2} \cdot c_2} = \frac{a_1 + a_2}{c_2}$$

$$= \sin(x+y).$$

It does work!!!

Example 1: Simplify each:

a. $\cos(\underset{A}{x}+\underset{B}{60^\circ}) =$ → We'll use $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$= \cos x \cos 60^\circ - \sin x \sin 60^\circ$

$= \cos x \cdot \frac{1}{2} - \sin x \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{\cos x - \sqrt{3}\sin x}{2}}$

b. $\sin\left(x+\frac{\pi}{4}\right)-\sin\left(x-\frac{\pi}{4}\right)$ → We'll use $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

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$\left[\cancel{\sin x \cos\frac{\pi}{4}} + \cos x \sin\frac{\pi}{4}\right] - \left[\cancel{\sin x \cos\frac{\pi}{4}} - \cos x \sin\frac{\pi}{4}\right]$

$= \cos x \sin\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} = 2\cos x \sin\frac{\pi}{4}$

$= \cancel{2}\cos x \cdot \frac{\sqrt{2}}{\cancel{2}} = \boxed{\sqrt{2}\cos x}$

Example 2: Given that $\tan(x) = 5$, evaluate $\tan\left(x + \frac{3\pi}{4}\right)$.

We'll use $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\Longrightarrow \tan\left(x + \frac{3\pi}{4}\right) = \frac{\tan x + \tan\left(\frac{3\pi}{4}\right)}{1 - \tan x \cdot \tan\left(\frac{3\pi}{4}\right)}$$

$$= \frac{5 + (-1)}{1 - 5 \cdot (-1)} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

Example 3: Simplify each.

We'll use $\sin(x-y) = \sin x \cos y - \cos x \sin y$

backwards

a. $\sin 10° \cos 55° - \sin 55° \cos 10°$

$= \sin 10° \cos 55° - \cos 10° \cdot \sin 55°$

$= \sin(10° - 55°) = \sin(-45°) = -\sin 45° = \boxed{-\frac{\sqrt{2}}{2}}$

b. $\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\sin\left(\frac{7\pi}{12}\right)$

We'll use $\cos(x-y) = \cos x \cos y + \sin x \sin y$

backwards

$= \cos\left(\frac{\pi}{12} - \frac{7\pi}{12}\right)$

$= \cos\left(\frac{-6\pi}{12}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

c. $\dfrac{\tan 40° + \tan 5°}{1 - \tan 40° \tan 5°} = \tan(40° + 5°)$

$= \tan 45° = 1$

Use $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

backwards

d. $\dfrac{\tan 80° - \tan 15°}{1 + \tan 80° \tan 15°} = \tan(80° - 15°)$

$= \tan(65°)$

Use $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

backwards

Example 4: Find the exact value of each. (Hint: use sum/difference formulas)

a. $\sin 15^0$

$30^\circ, 45^\circ, 60^\circ$

$\underbrace{30^\circ, 45^\circ}$ difference $= 15^\circ$

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$\sin(45^\circ - 30^\circ)$

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$\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

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$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$

30°	45°	60°	90°
$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

$\underbrace{\frac{\pi}{4} \quad \frac{\pi}{3}}_{\text{check}}$

b. $\cos\left(\frac{7\pi}{12}\right)$ → think $\frac{7\pi}{12} = 105^\circ = \underbrace{45^\circ + 60^\circ}$

→ the sum $\frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$

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$\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

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$\cos\frac{\pi}{4} \cdot \cos\frac{\pi}{3} - \sin\frac{\pi}{4} \sin\frac{\pi}{3}$

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$\frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$

30°	45°	60°	90°	120°	135°	150°	180°
$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

$\underbrace{\frac{\pi}{6} \quad \frac{\pi}{4}}_{\text{their sum}}$

c. $\tan\left(\frac{5\pi}{12}\right)$ $\longrightarrow$ $\frac{5\pi}{12} = 75^\circ = 30^\circ + 45^\circ$, $\quad \frac{4\cdot\pi}{4\cdot 6} + \frac{\pi\cdot 6}{4\cdot 6} = \frac{10\pi}{24} = \frac{5\pi}{12}$

$$= \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

$$= \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\cdot\tan\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{3}}{3} + 1\cdot\frac{3}{3}}{\frac{3}{3}\cdot 1 - \frac{\sqrt{3}}{3}\cdot 1} = \frac{\frac{\sqrt{3}+3}{\cancel{3}}}{\frac{3-\sqrt{3}}{\cancel{3}}} = \frac{\sqrt{3}+3}{3-\sqrt{3}}$$

Very Important $\longrightarrow$

$$= \frac{(\sqrt{3}+3)}{3-\sqrt{3}}\cdot\frac{(3+\sqrt{3})}{(3+\sqrt{3})} \overset{\text{FOIL}}{=} \frac{3\sqrt{3} + (\sqrt{3})^2 + 9 + 3\sqrt{3}}{3^2 - (\sqrt{3})^2}$$

$(a-b)(a+b) = a^2 - b^2$

(a square eliminates the radical $\sqrt{\cdot}$)

$$= \frac{6\sqrt{3} + 12}{9-3} = \frac{6\sqrt{3}+12}{6} = \frac{\cancel{6}(\sqrt{3}+2)}{\cancel{6}}$$

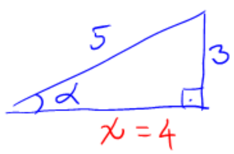
$$= \boxed{\sqrt{3} + 2}$$

To be continued on Thursday, ~~04~~/07

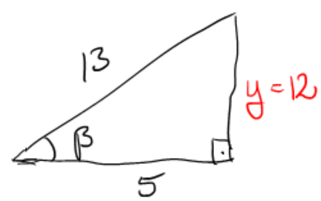
both angles in Quadrant I

Example 5: Suppose that $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$ where $0 < \alpha < \beta < \frac{\pi}{2}$. Find each of these:

$\sin\alpha = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$



$\cos\beta = \frac{5}{13} \Rightarrow \sin\beta = \frac{12}{13}$



a. $\sin(\alpha + \beta)$

$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$= \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13}$$

$$= \frac{15}{65} + \frac{48}{65} = \boxed{\frac{63}{65}}$$

b. $\cos(\alpha - \beta)$

$$\cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13}$$

$$= \frac{20}{65} + \frac{36}{65} = \boxed{\frac{56}{65}}$$

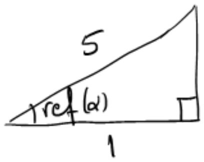
Think: $\begin{cases} \pi < \alpha < 2\pi \\ \cos\alpha = \frac{1}{5} \text{ positive} \end{cases}$ $\Rightarrow \alpha$ in Quadrant IV

$\begin{cases} \pi < \beta < 2\pi \\ \tan\beta = -\frac{7}{6} = \text{negative} \end{cases}$ $\Rightarrow \beta$ in Quadrant IV

Example 6: Suppose $\cos\alpha = \dfrac{1}{5}$ and $\tan\beta = -\dfrac{7}{6}$ where $\pi < \alpha, \beta < 2\pi$. Find

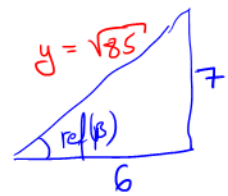
$\Rightarrow \sin\alpha = \text{negative}$

$\Rightarrow \sin\beta = \text{negative}, \cos\beta = \text{positive}$



$x = \sqrt{24} = 2\sqrt{6}$

$\Rightarrow \sin\alpha = \dfrac{-2\sqrt{6}}{5}$



$y = \sqrt{6^2 + 7^2} = \sqrt{85}$

$\sin\beta = -\dfrac{7}{\sqrt{85}} = \dfrac{-7\sqrt{85}}{85}$

$\cos\beta = \dfrac{6}{\sqrt{85}} = \dfrac{6\sqrt{85}}{85}$

a. $\cos(\alpha + \beta)$

$\parallel$

$\cos\alpha \cdot \cos\beta - \sin\alpha \sin\beta$

$= \dfrac{1}{5} \cdot \dfrac{6\sqrt{85}}{85} - \left(\dfrac{-2\sqrt{6}}{5}\right)\left(\dfrac{-7\sqrt{85}}{85}\right)$

$= \dfrac{6\sqrt{85}}{5 \cdot 85} - \dfrac{14\sqrt{6}\cdot\sqrt{85}}{5 \cdot 85}$

b. $\tan(\alpha + \beta)$

$= \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

Note: $\tan\alpha = \dfrac{-2\sqrt{6}}{1} = -2\sqrt{6}$

$\tan\beta = -\dfrac{7}{6}$

$= \dfrac{\frac{6}{6} \cdot \frac{-2\sqrt{6}}{1} - \frac{7}{6}}{\frac{6}{6}\cdot 1 - (-2\sqrt{6})\left(-\frac{7}{6}\right)} = \dfrac{\frac{-12\sqrt{6} - 7}{6}}{\frac{6}{6} - \frac{14\sqrt{6}}{6}} = \dfrac{\frac{-12\sqrt{6} - 7}{6}}{\frac{6 - 14\sqrt{6}}{6}}$

$= \dfrac{-12\sqrt{6} - 7}{\cancel{6}} \cdot \dfrac{\cancel{6}}{6 - 14\sqrt{6}} = \dfrac{(-1)}{(-1)}\dfrac{-12\sqrt{6} - 7}{6 - 14\sqrt{6}} = \boxed{\dfrac{12\sqrt{6} + 7}{14\sqrt{6} - 6}}$