Popper # 19

(1) $\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \cos\left(2 \times \frac{\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

A. 0 B. 1 C. $\frac{1}{2}$ D. $\frac{\sqrt{2}}{2}$ E. none

Given $\sin(x) = \frac{3}{5}$, $\cos(y) = \frac{12}{13}$, $0 < x < 90°$, $270° < y < 360°$.

$\cos x = \frac{4}{5}$ $\sin(y) = -\frac{5}{13}$

(2) $\sin(x+y) = \sin x \cos y + \cos x \sin y = \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \left(-\frac{5}{13}\right) = \frac{16}{65}$

A. $\frac{36}{65}$ B. $\frac{16}{65}$ C. $\frac{-16}{65}$ D. $\frac{-20}{65}$

(3) $\tan\left(\frac{y}{2}\right) = \frac{1-\cos y}{\sin y} = \frac{1 - \frac{12}{13}}{\frac{-5}{13}} = \frac{\frac{1}{\cancel{13}}}{\frac{-5}{\cancel{13}}} = -\frac{1}{5}$

A. $-\frac{12}{5}$ B. $-\frac{1}{5}$ C. $\frac{5}{13}$ D. none

(4) $\cos(2x) = \cos^2 x - \sin^2 x = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

A. $\frac{7}{25}$ B. $\frac{18}{25}$ C. $\frac{-7}{25}$ D. none

(5) Bubble A

(6) Bubble B

Recall:

- Linear Equation:

$$2x+3=9 \quad \longleftarrow \quad \text{Solve it!}$$
$$\quad -3 \quad -3$$

"Goal: Leave the x alone."

$$\frac{2x}{2}=\frac{6}{2} \implies \boxed{x=3}$$

- Trigonometric Equation:

$$\sin(x)=\frac{1}{2} \quad \longleftarrow \quad \text{Find all } x.$$

"Goal: $x=?$". Look for solution in the unit circle first.

$$\implies x=\frac{\pi}{6} \quad \text{or} \quad x=\frac{5\pi}{6} \quad \text{in the first period}$$

Trigonometric Functions are periodic

$\implies$ we get infinitely many solutions when possible.

$$x=\frac{\pi}{6}+2\pi\cdot k \quad \text{or} \quad x=\frac{5\pi}{6}+2\pi\cdot k, \quad k \text{ integer!}$$

Linear Equation: $2x + 3 = 9 \leftarrow$ Solve!
$-3 \quad -3$
"Goal is to leave x alone"
$\frac{2x}{2} = \frac{6}{2} \implies \boxed{x = 3}$

Section 6.3 - Solving Trigonometric Equations

Next, we'll use all of the tools we've covered in our study of trigonometry to solve some equations. An equation that involves a trigonometric function is called a trigonometric equation. Since trigonometric functions are periodic, there may be infinitely solutions to some trigonometric equations.

• Trigonometric Equation ← Unit Circle

Let's say we want to solve the equation: $\sin(x) = \frac{1}{2}$

Ask yourself: Which angle(s) have sine $= \frac{1}{2}$?

The first angles that come to mind are: $\boxed{x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}}$. ← Answer

in one period

$x = 30°$ or $150°$ (convert in radians)

Remember that the period of the sine function is 2π; sine function repeats itself after each rotation.

The solutions of unit circle repeat themselves in every periodic rotation.

Therefore, the solutions of the equation are: $x = \underbrace{\frac{\pi}{6}}_{\text{unit circle}} + \underbrace{2k\pi}_{2\pi \cdot k}$, $x = \underbrace{\frac{5\pi}{6}}_{\text{unit circle}} + \underbrace{2k\pi}_{2\pi \cdot k}$, where k is any integer.

Recall: For sine and cosine functions, the period is 2π. For tangent and cotangent functions, the period is π.

Do NOT FORGET:

General (ALL) Solutions = $\underbrace{\text{Special Solutions}}_{\text{unit circle}}$ + Period · k

one rotation

Example 1: a) Solve the equation in the interval $[0,2\pi)$: $2\cos x = -1$

$2\cos x = -1$

Q.II Q.III

$\cos x = -\frac{1}{2} \implies x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

period $= 2\pi$

b) Find all solutions to the equation: $2\cos x = -1$

From part (a), $\cos x = -\frac{1}{2} \implies x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

all $\boxed{x = \frac{2\pi}{3} + 2\pi \cdot k \quad \text{or} \quad x = \frac{4\pi}{3} + 2\pi \cdot k,}$ k integer

one period

Example 2: a) Solve the equation in the interval $[0,\pi)$: $\tan x = -1$

$\tan x = -1$

From 0 to π,

only $\boxed{x = \frac{3\pi}{4}}$ in Quadrant II, gives $\tan\left(\frac{3\pi}{4}\right) = -1$

b) Find all solutions to the equation: $\tan x = -1$

period $= \pi$

$\implies \boxed{x = \frac{3\pi}{4} + \pi \cdot k,}$ k integer

Example 3: Solve the equation in the interval $[0,\pi)$: $2\sin(2x)=1$

one period

$2\sin(2x) = 1$

$\sin(2x) = \frac{1}{2} \Rightarrow \frac{2x}{2} = \frac{\frac{\pi}{6}}{2}$ or $\frac{2x}{2} = \frac{\frac{5\pi}{6}}{2}$

period $= \frac{2\pi}{2} = \pi$

$\Rightarrow \boxed{x = \frac{\pi}{12} \text{ or } x = \frac{5\pi}{12}}$

Example 4: Solve the equation in the interval $[0,2\pi)$: $\csc^2 x = 4$ $\leftarrow$ In one period

$\csc^2 x = 4 \iff \csc x = +2$ or $\csc x = -2$

$\csc x = \pm 2$

$\frac{1}{\sin x} = 2$ $\qquad \frac{1}{\sin x} = -2$

$\sin x = \frac{1}{2}$ $\qquad \sin x = \frac{-1}{2}$

$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

Thursday,

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Example 5: Find all solutions to the equation: $\cos(2x)=0$

need to find solutions in one period first and add repetitions of periods.

$\rightarrow \cos(2x) = 0 \Rightarrow \frac{2x}{2} = \frac{\frac{\pi}{2}}{2}$ or $\frac{2x}{2} = \frac{\frac{3\pi}{2}}{2}$

period $= \frac{2\pi}{2} = \pi$

$x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$

All solutions: $\boxed{\begin{array}{l} x = \frac{\pi}{4} + \pi \cdot k, \quad k \text{ integer.} \\ x = \frac{3\pi}{4} + \pi \cdot k \end{array}}$

To be continued on Thursday, 04/14.

one period

Example 6: Solve the equation in the interval $[0,2\pi)$: $2\sin^2 x - 5\sin x - 3 = 0$

$2\sin^2 x - 5\sin x - 3 = 0$

$\underbrace{(\sin x - 3)}_{0}\underbrace{(2\sin x + 1)}_{0} = 0$

$\cancel{\sin x - 3 = 0}$ or $2\sin x + 1 = 0$

$\cancel{\sin x = 3}$ $\quad 2\sin x = -1$

Can't happen $\quad \sin x = -\frac{1}{2}$

$\Longrightarrow$ $\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$ one period

Example 7: Solve the equation in the interval $[0,2\pi)$: $\cos^2 x - 3\sin x - 3 = 0$

$\underbrace{\cos^2 x} - 3\sin x - 3 = 0$ (Transform in an equation with just one trig. function)

$1 - \sin^2 x - 3\sin x - 3 = 0$

$-\sin^2 x - 3\sin x - 2 = 0$

$\sin^2 x + 3\sin x + 2 = 0$

$\underbrace{(\sin x + 1)}_{0}$ or $\underbrace{(\sin x + 2)}_{0} = 0$

$\sin x + 1 = 0$ or $\cancel{\sin x + 2 = 0}$

$\sin x = -1$ $\quad \cancel{\sin x = -2}$

$\boxed{x = \frac{3\pi}{2}}$ $\quad$ Can't happen

Example 8: Solve the equation in the interval $[0,2\pi)$: $\cos(2x) = 5\sin^2 x - \cos^2 x$

$\underbrace{\cos(2x)}_{2\cos^2 x - 1} = 5\underbrace{\sin^2 x}_{1-\cos^2 x} - \cos^2 x$ (Always, keep just one trig. expression.)

(We'll do everything with cosx)

$\Longrightarrow 2\cos^2 x - 1 = 5(1 - \cos^2 x) - \cos^2 x$

$2\cos^2 x - 1 = 5 - 5\cos^2 x - \cos^2 x = 5 - 6\cos^2 x$

$+6\cos^2 x \quad +1 \qquad\qquad +1 \quad +6\cos^2 x$

$8\cos^2 x = 6 \Longleftrightarrow \cos^2 x = \frac{6}{8} \Longleftrightarrow \cos^2 x = \frac{3}{4}$

$\cos x = \pm\sqrt{\frac{3}{4}} \longrightarrow \cos x = +\frac{\sqrt{3}}{2}$ or $\cos x = -\frac{\sqrt{3}}{2}$

$\boxed{x = \frac{\pi}{6}, \frac{11\pi}{6} \quad \text{or} \quad x = \frac{5\pi}{6}, \frac{7\pi}{6}}$

 one period

Example 8: Solve the equation in the interval $[0,2\pi)$: $\cos(2x) = 5\sin^2 x - \cos^2 x$

- Transform into an expression with only one trigonometric function if possible.
- We'll do everything with sin(x)

$$\underbrace{\cos(2x)} = 5\sin^2 x - \underbrace{\cos^2 x}$$

$$1-2\sin^2 x = 5\sin^2 x - (1-\sin^2 x)$$

$$1 - \cancel{2\sin^2 x} = 5\sin^2 x - \cancel{1} + \sin^2 x$$

$$+1 \quad + \cancel{2\sin^2 x} \qquad\qquad +\cancel{1} \quad +2\sin^2 x$$

$$2 = 8\sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{2}{8} = \frac{1}{4} \Rightarrow \sin x = \pm\frac{1}{2}$$

$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -\frac{1}{2}$$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{or} \quad x = \frac{7\pi}{6}, \frac{11\pi}{6}}$$

Example 9: Find all solutions to the equation: $\sin^2 x \cos x = \cos x$

(there is a common factor, bring in one side)

$\sin^2 x \cdot \cos x = \cdot \cos x$

$\sin^2 x \cos x - \cos x = 0$

$\underbrace{\cos x}_{0} \underbrace{(\sin^2 x - 1)}_{0} = 0$

$\cos x = 0$ or $\sin^2 x = 1 \Rightarrow \sin x = \pm 1$

$\underbrace{x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \frac{\pi}{2}, \frac{3\pi}{2}}_{\text{in one period}}$

All Solutions:

$$\boxed{\begin{aligned} x &= \frac{\pi}{2} + 2\pi \cdot k, \\ x &= \frac{3\pi}{2} + 2\pi \cdot k, \end{aligned}}$$

k integer

(use $\sec^2 x = 1 + \tan^2 x$)

Example 10: Find all solutions: $\sec^2 x + 2\tan x = 0$

$\underbrace{\sec^2 x}_{} + 2\tan x = 0$

$\tan^2 x + 1 + 2\tan x = 0 \iff (\tan x + 1)(\tan x + 1) = 0$

i.e. $\tan x + 1 = 0$

$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}$ in one period $= \pi$

$\Rightarrow \boxed{x = \frac{3\pi}{4} + \pi \cdot k,}$ k integer.

Another version $\Rightarrow \tan x = -1 \Rightarrow x = -\frac{\pi}{4} \leftarrow$ Solution

Note that $-\frac{\pi}{4} = \frac{3\pi}{4} + \pi \cdot (-1)$

$\Rightarrow \boxed{x = -\frac{\pi}{4} + \pi \cdot k,}$ k integer

Both equivalent

there are two periodical intervals

Example 11: Solve the equation in the **interval** $[0,2)$: $\cot(\pi x) = -1$

$\cot(\pi x) = -1$

period $= \frac{\pi}{\pi} = 1$

$\frac{\cancel{\pi} x}{\cancel{\pi}} = \frac{\frac{3\cancel{\pi}}{4}}{\cancel{\pi}}$ ⟵ over one period

$x = \frac{3}{4} \Rightarrow x = \frac{3}{4} + 1 = \frac{7}{4}$

next period.

$$\Rightarrow \boxed{x = \frac{3}{4}, \frac{7}{4}}$$

Example 12: Find all solutions of the equation in the interval $[0,4\pi)$: $2\sin\left(\frac{x}{2}\right) = 1$

one period

$2\sin\left(\frac{x}{2}\right) = 1$

$\Rightarrow \sin\left(\frac{x}{2}\right) = \frac{1}{2} \Rightarrow$ Think, in one full rotation,

period $= \frac{2\pi}{1/2} = 4\pi$

$2 \times \frac{x}{2} = \frac{\pi}{6} \times 2$ or $2 \times \frac{x}{2} = \frac{5\pi}{6} \times 2$

$x = \frac{\cancel{2}\pi}{\cancel{6}} = \frac{\pi}{3}$, $x = \frac{\overset{5}{\cancel{10}}\pi}{\underset{3}{\cancel{6}}} = \frac{5\pi}{3}$

$$\Rightarrow \boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$$

Always, use identities (if possible) to simplify!

exercise Example 13: Find all solutions of the equation in the interval $[0,2\pi)$: $\sec(x+2\pi)=2$

period $=2\pi$

Hence, $\sec(\underbrace{x+2\pi}_{\text{period}})=\sec(x)$

Thus, $\sec(x)=2$

$$\frac{1}{\cos x}=2$$

$$\cos x=\frac{1}{2} \Longrightarrow \boxed{x=\frac{\pi}{3},\ \frac{5\pi}{3}}$$

exercise

Example 14: Find all solutions of the equation in the interval $[0,\pi)$: $2\sin\left(2x-\frac{3\pi}{2}\right)=\sqrt{2}$

one period

$$\Rightarrow \quad 2\sin\left(2x-\frac{3\pi}{2}\right)=\sqrt{2}$$

$$\sin\left(2x-\frac{3\pi}{2}\right)=\frac{\sqrt{2}}{2}$$

$\frac{\pi}{4}$ or $\frac{3\pi}{4}$

(there is no identity to apply, hence go to unit circle.)

In one full rotation, this expression

$$2x-\frac{3\pi}{2}=\frac{\pi}{4} \quad \text{or} \quad 2x-\frac{3\pi}{2}=\frac{3\pi}{4}$$

$$2x=\frac{\pi}{4}+\frac{3\pi}{2}\cdot\frac{2}{2} \qquad 2x=\frac{3\pi}{4}+\frac{3\pi}{2}\cdot\frac{2}{2}$$

$$\frac{2x}{2}=\frac{\frac{7\pi}{4}}{2} \qquad \frac{2x}{2}=\frac{\frac{9\pi}{4}}{2}$$

$$\rightarrow \boxed{x=\frac{7\pi}{8}} \leftarrow \text{Answer.} \qquad x=\frac{9\pi}{8} \text{ not in } [0,\pi)$$