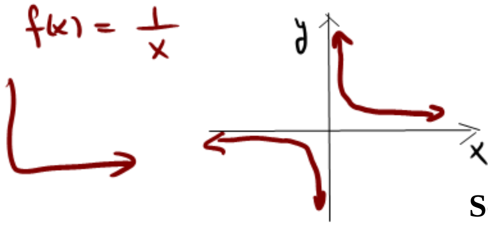
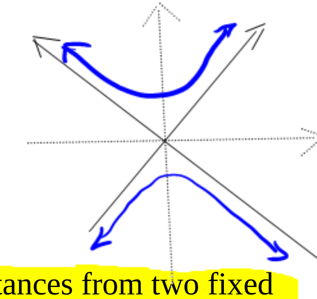
Think of $f(x)=\frac{1}{x}$



Rotate like 45° counterclockwise

+45°



You get a "Vertical Hyperbola"

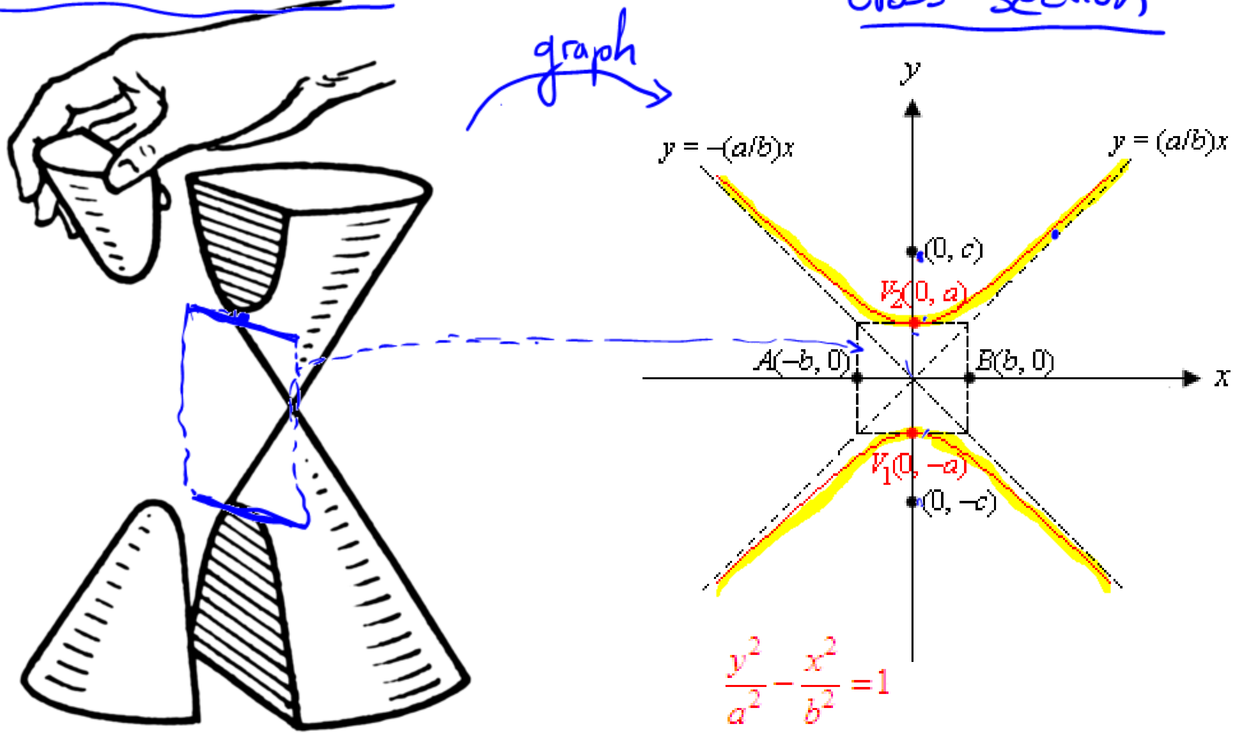
Section 8.3 Hyperbolas

Definition: A *hyperbola* is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a *focus* (plural = *foci*).

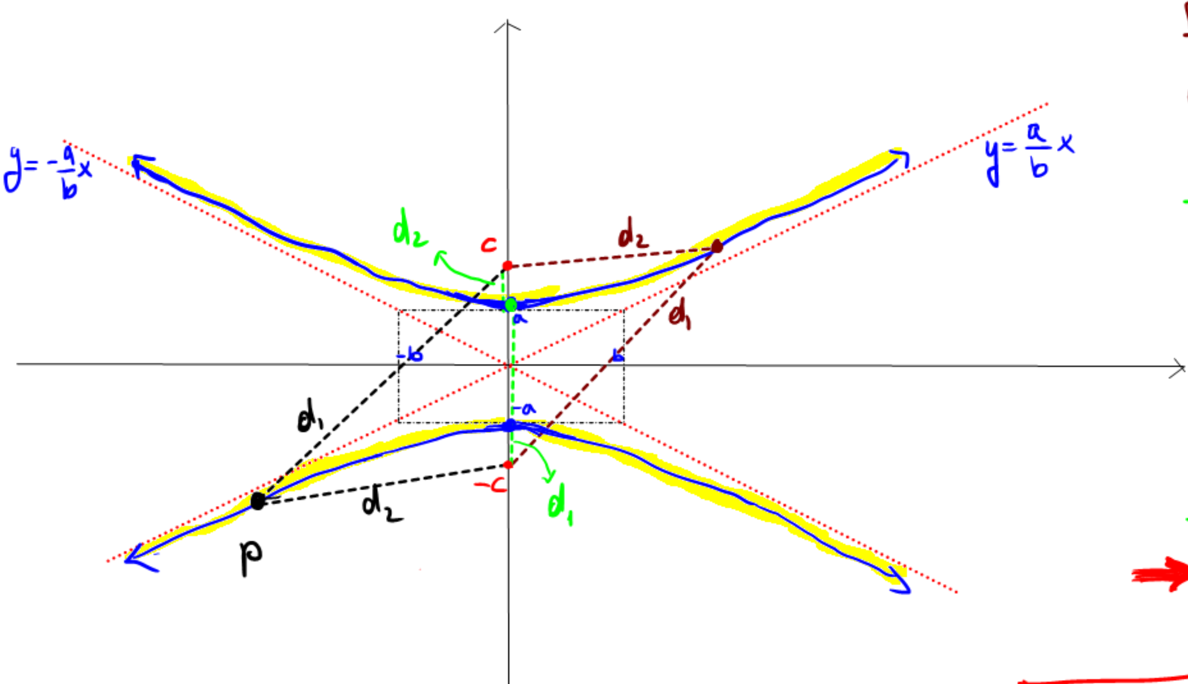
The *focal axis* is the line passing through the foci.

Visualize how it is built!

Cross-section



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$



By definition

$d_1 - d_2 = \underbrace{\text{fixed}}_{2a}$

$d_1 - d_2 = \text{fixed}$

$d_1 = c + a$

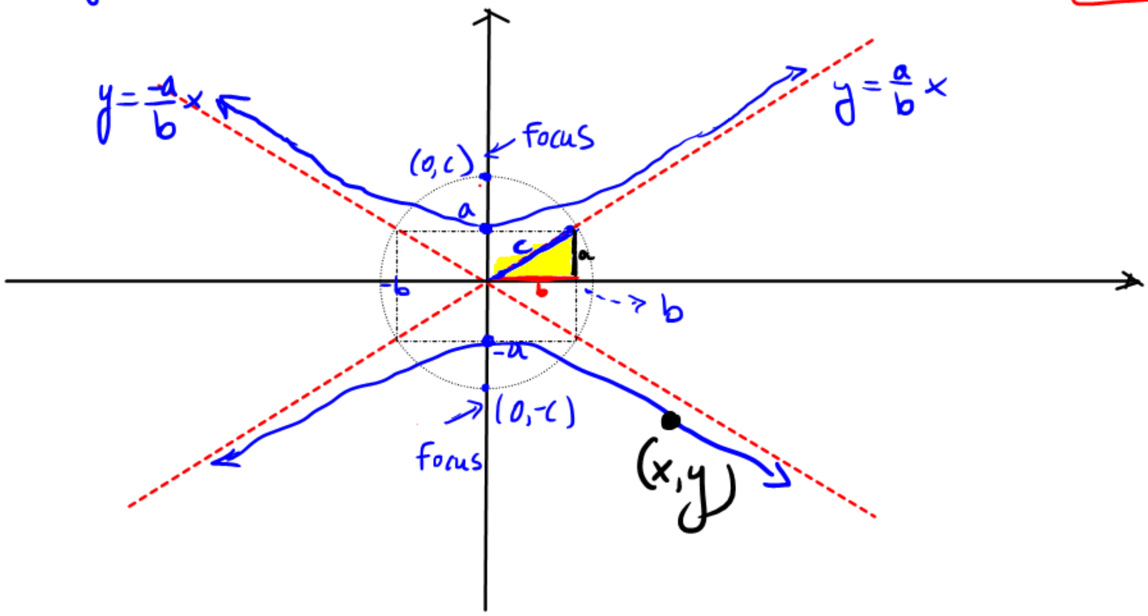
$d_2 = c - a$

$d_1 - d_2 = 2a$

$\Rightarrow d_1 - d_2 = 2a$

$\Rightarrow$ By using similar strategy as in parabolas or ellipses, we get $\boxed{c^2 = a^2 + b^2}$

The beauty of hyperbolas is that the foci and the corners of the rectangle are in a circle. This occurs because $\boxed{c^2 = a^2+b^2}$



For any fixed point (x,y) on hyperbola above doing the difference of distances of this point to the foci very similarly as we did for ellipse, but here we have $c^2=a^2+b^2$, we deduce

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

Standard form: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

begins with $\boxed{\dfrac{y^2}{a^2}}$

Basic "vertical" hyperbola:

Equation: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

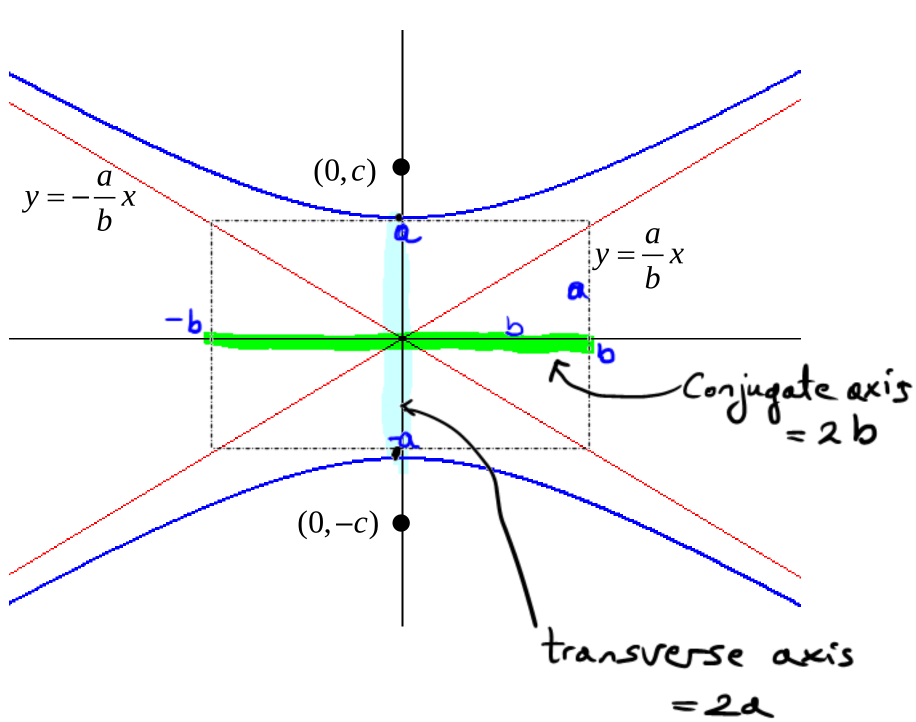
Asymptotes: $y = \pm\dfrac{a}{b}x$

always

Foci: $(0, \pm c)$, where $\boxed{c^2 = a^2 + b^2}$

Vertices: $(0, \pm a)$

Eccentricity: $\dfrac{c}{a}$ (>1)



begins with $\dfrac{x^2}{a^2}$.

Basic "horizontal" hyperbola:

Equation: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

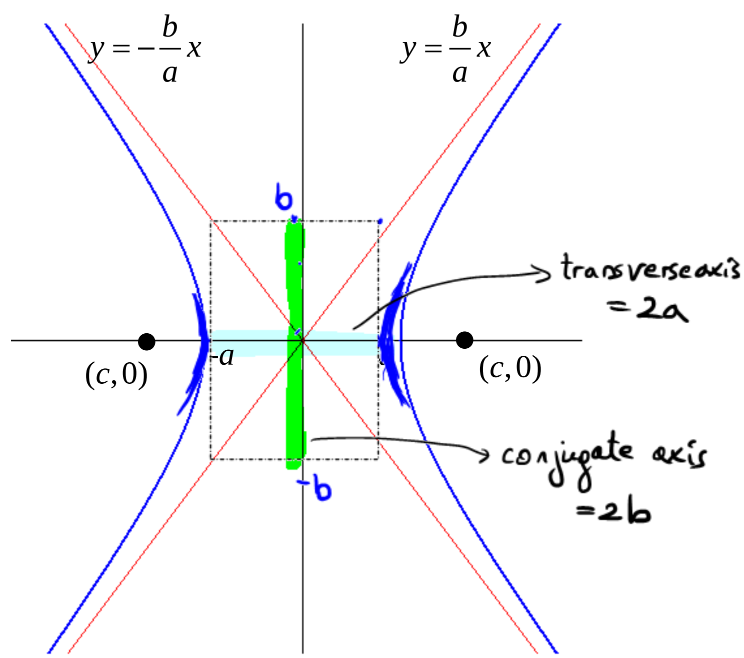
Asymptotes: $y = \pm\dfrac{b}{a}x$

always

Foci: $(\pm c, 0)$, where $\boxed{c^2 = a^2 + b^2}$

Vertices: $(\pm a, 0)$

Eccentricity: $\dfrac{c}{a}$ (>1)

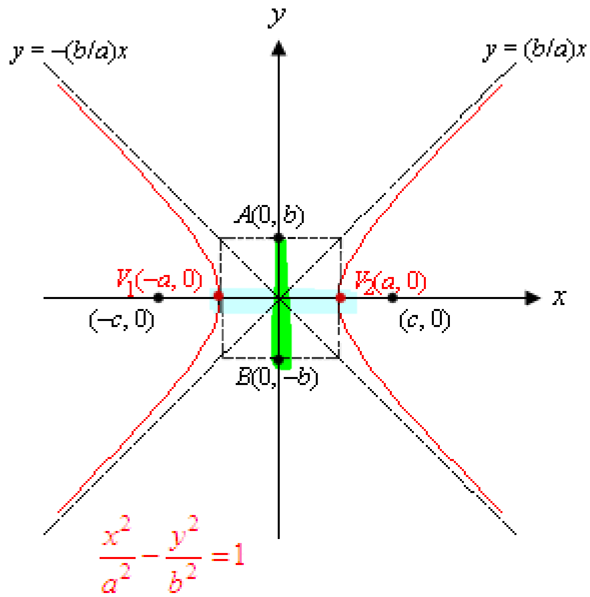


*Note: The **transverse axis** is the line segment joining the two vertices. The **conjugate axis** is the line segment perpendicular to the transverse axis, passing through the center and extending a distance b on either side of the center. (These terms will make more sense after we do the graphing examples.)*

Never forget: Hyperbola curves lie in between two intersecting lines opposite to each other.

Details about conjugate and transverse axis.

The conjugate axis $=2b$ of the hyperbola is the line segment through the center of the hyperbola and perpendicular to the transverse axis $=2a$ with endpoints $(0,-b)$ and $(0,b)$.



Center: $(0,0)$

Foci: $(-c,0)$ and $(c,0)$, where $c^2=a^2+b^2$

Vertices: $V_1(-a,0)$ and $V_2(a,0)$

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

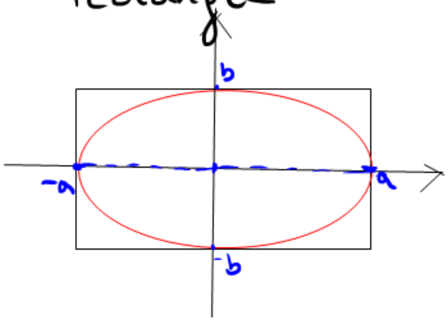
The eccentricity of a hyperbola is given by the formula $e=\frac{c}{a}$.

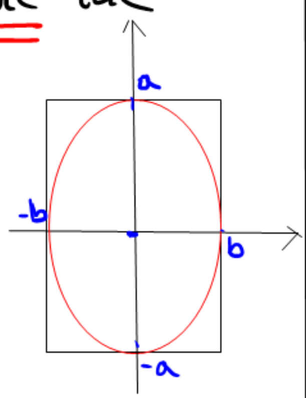
The lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ are slant asymptotes for the hyperbola since it can be shown that as $|x|$ becomes large, $y\to\pm\frac{b}{a}x$.

Ellipses Vs. Hyperbolas

Ellipses

- by definition $d_1 + d_2 = 2a$
- Foci Equation $c^2 = a^2 - b^2$
  (foci are between vertices)
- Horizontal Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
  - Vertices $(a,0)$, $(-a,0)$
- Vertical Ellipse $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
  - Vertices $(0,a)$, $(0,-a)$
- Major axis $= 2a$
  (connects the vertices)
- Minor axis $= 2b$
- No Slant asymptotes
- Ellipse is <u>inside</u> the rectangle



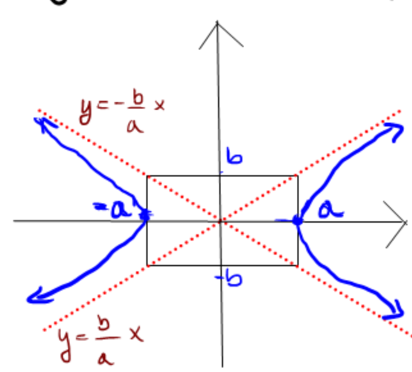


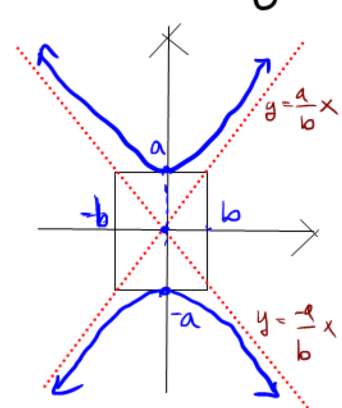
Hyperbolas

- by definition $|d_1 - d_2| = 2a$.
- Foci Equation $c^2 = a^2 + b^2$
  (foci are beyond vertices)
- Horizontal Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
  - Vertices $(a,0)$, $(-a,0)$
- Vertical Hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
  - Vertices $(0,a)$, $(0,-a)$
- Transverse axis $= 2a$
  (connects the vertices)
- Conjugate axis $= 2b$
- slant asymptotes

  $y = \pm \frac{b}{a}x$ or $y = \pm \frac{a}{b}x$

  Horizontal Vertical
- Hyperbola is <u>beyond</u> the rectangle





To be continued on Thursday, 2/18.

Graphing hyperbolas: Let's demonstrate by doing example 1.

To graph a hyperbola with center at the origin:

Just know how to begin!!!

- Rearrange into the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

- Decide if it's a "horizontal" or "vertical" hyperbola.
  - if x^2 comes first, it's horizontal (vertices are on *x*-axis).
  - If y^2 comes first, it's vertical (vertices are on *y*-axis).
- Use the square root of the number under x^2 to determine how far to measure in *x*-direction.

- Use the square root of the number under y^2 to determine how far to measure in *y*-direction.

- Draw a box with these measurements.

- Draw diagonals through the box. These are the asymptotes. Use the dimensions of the box to determine the slope and write the equations of the asymptotes.

- Put the vertices at the edge of the box on the correct axis. Then draw a hyperbola, making sure it approaches the asymptotes smoothly.

- $\boxed{c^2=a^2+b^2}$ where a^2 and b^2 are the denominators.

- The foci are located *c* units from the center, on the same axis as the vertices.

When graphing hyperbolas, you will need to find the orientation, center, values for a, b and c, lengths of transverse and converse axes, vertices, foci, equations of the asymptotes, and eccentricity.

→ let's begin:

Example 1: Find all relevant information and graph $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

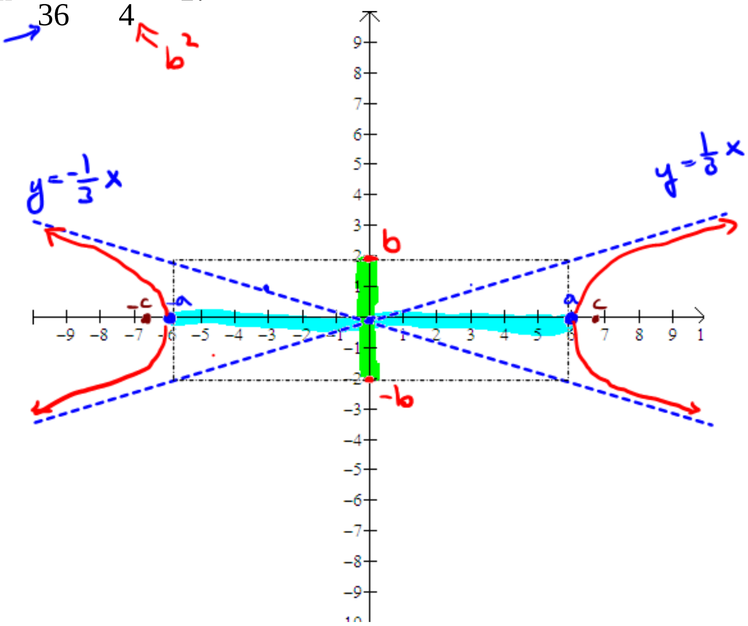
$\to$ horizontal

$a^2 \to 36$, $4 \leftarrow b^2$

$a^2 = 36 \Rightarrow a = 6$

$b^2 = 4 \Rightarrow b = 2$

$c^2 = a^2 + b^2 = 40 \Rightarrow c = \sqrt{40} = 2\sqrt{10}$



Vertices: $(6,0)$, $(-6,0)$

Foci: $(2\sqrt{10}, 0)$, $(-2\sqrt{10}, 0)$

Eccentricity: $\frac{c}{a} = \frac{2\sqrt{10}}{6} =$

Transverse Axis: = The segment joining vertices

Length of transverse axis:

$2 \cdot a = 2 \cdot 6 = 12$

Conjugate axis: = The segment joining the "b" points.

Length of conjugate axis:

$2 \cdot b = 2 \cdot 2 = 4$

Slant Asymptotes: $y = \frac{b}{a}x = \frac{2}{6}x = \frac{1}{3}x$

$y = -\frac{b}{a}x = -\frac{1}{3}x$

vertical hyperbola with center (0,0)

Example 2: Find all relevant information and graph $\frac{y^2}{4}-\frac{x^2}{9}=1$.

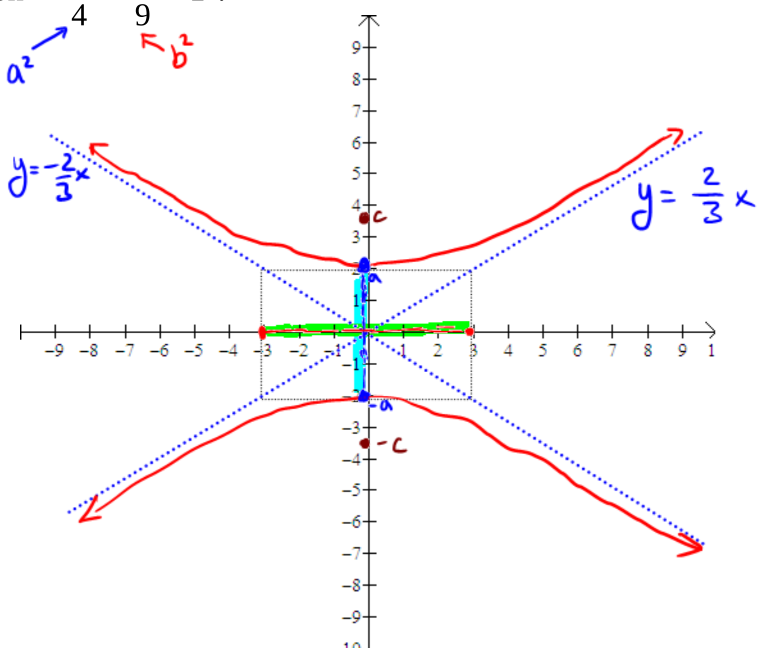
a^2 b^2

$a^2=4 \Rightarrow a=\pm 2$ (vertices)

$b^2=9 \Rightarrow b=\pm 3$

$c^2=a^2+b^2=4+9=13$

$\Rightarrow c=\pm\sqrt{13}$



Vertices: $(0,2)$, $(0,-2)$

Foci: $(0,\sqrt{13})$, $(0,-\sqrt{13})$

Eccentricity: $\frac{c}{a}=\frac{\sqrt{13}}{2}$

Transverse Axis: the segment joining vertices

Length of transverse axis: $2a=2\cdot 2=4$

Conjugate axis: the segment joining "b"

Length of conjugate axis: $2b=2\cdot 3=6$

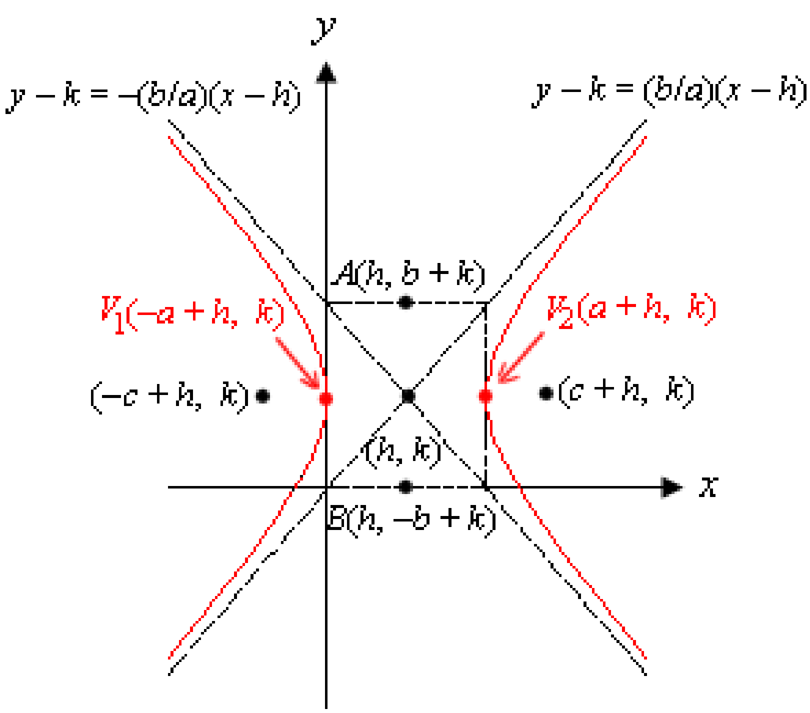
Slant Asymptotes: $y=\pm\frac{a}{b}x=\pm\frac{2}{3}x$

The equation of a hyperbola with center not at the origin: **Center: (h, k)**

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

To graph a hyperbola with center not at the origin:

- Rearrange (complete the square if necessary) to look like $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.
- Start at the center (h,k) and then graph it as before.
- To write down the equations of the asymptotes, start with the equations of the asymptotes for the similar hyperbola with center at the origin. Then replace *x* with $x-h$ and replace *y* with $y-k$.



$\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ shift h units horizontally, shift k units vertically $\longrightarrow$ $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

center $(0,0)$ $\longrightarrow$ center (h,k)

- All the features shift accordingly.

The following list reflects the changes in translating the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$:

Center: The point $(0,0)$ changes to the point (h,k).

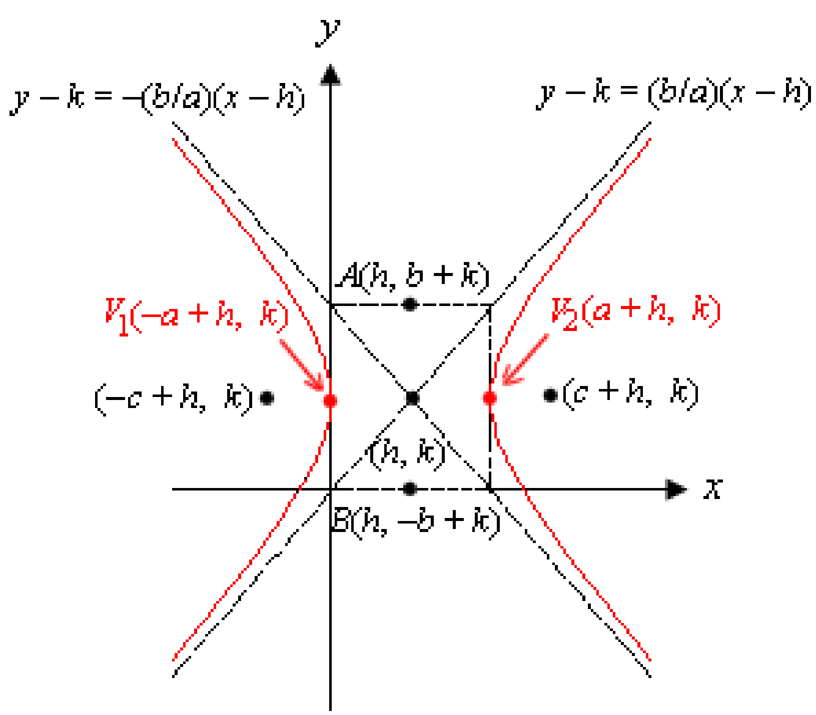
Foci: The foci change from the points $(-c,0)$ and $(c,0)$ to the points $(-c+h,k)$ and $(c+h,k)$, where $c^2=a^2+b^2$.

Vertices: The vertices change from the points $(-a,0)$ and $(a,0)$ to the points $(-a+h,k)$ and $(a+h,k)$.

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

Equations of the Asymptotes: The lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ change to the lines $y-k=\frac{b}{a}(x-h)$ and $y-k=-\frac{b}{a}(x-h)$.



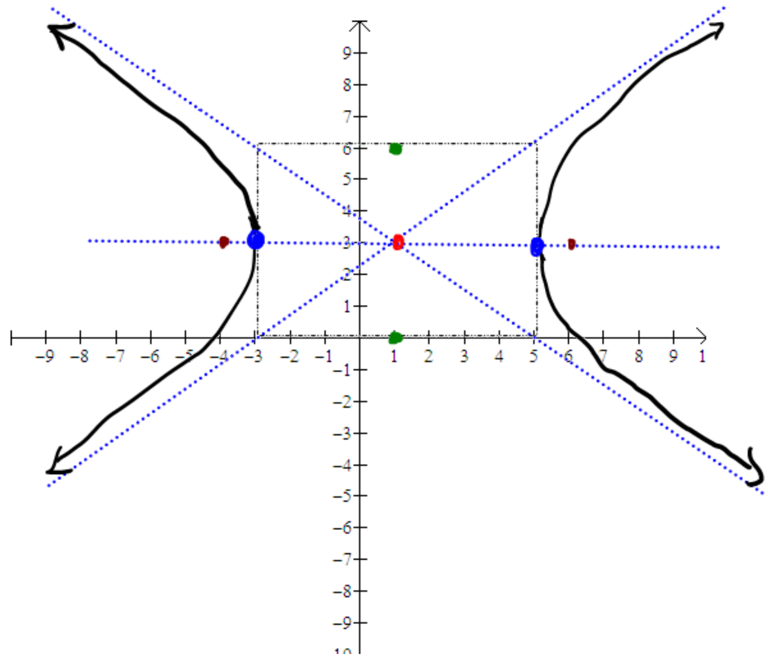
(look at the next page how to bring in standard form:

Example 3: Write the equation in standard form, find all relevant information and graph

$9x^2 - 16y^2 - 18x + 96y = 279$.

Standard form $\rightarrow \dfrac{(x-1)^2}{16} - \dfrac{(y-3)^2}{9} = 1$

$a^2 \nearrow 16$, $\quad 9 \nwarrow b^2$



Horizontal hyperbola with center (0,0)	• Horizontal hyperbola shifted 1 right, 3 up center (1,3) ✓
$a^2=16 \Rightarrow a=4$ $(-4,0),(4,0)$	• Vertices shifted $(-3,3)$ ✓, $(5,3)$ ✓
$2a=2\cdot4=8$	• Transverse axis $= 2a = 8$
$c^2=a^2+b^2=25$ $c=5$ $(-5,0),(5,0)$	• Foci Coordinates shifted $(-4,3)$ ✓, $(6,3)$ ✓
$b^2=9 \Rightarrow b=3$ $2b=2\cdot3=6$ $(0,-3),(0,3)$	• Conjugate axis $=2b=6$ shifted $(1,0)$ ✓, $(1,6)$ ✓
$y=\frac{b}{a}x=\frac{3}{4}x$ $y=-\frac{b}{a}x=\frac{3}{4}x$	• Slant Asymptotes shifted $y-3=\frac{3}{4}(x-1)$, $y-3=-\frac{3}{4}(x-1)$
$e=\frac{c}{a}$	• Eccentricity $e=\frac{c}{a}=\frac{5}{4}=1.25$

$\Rightarrow$ Draw the rectangle. Then draw diagonals of rectangle and extend. Diagonals are the slant asymptotes

$$9x^2 - 16y^2 - 18x + 96y = 279$$

Likely terms

$$(9x^2 - 18x) + (-16y^2 + 96y) = 279$$

Factor coefficients and complete square

$$9(x^2 - 2x + 1) - 16(y^2 - 6y + 9) = 279 + 9 \cdot 1 - 16 \cdot 9$$

$$\left(\tfrac{2}{2} = 1\right)^2 \qquad \left(\tfrac{6}{2} = 3\right)^2$$

Rewrite

$$9(x-1)^2 - 16(y-3)^2 = 144$$

Divide by 144 both sides

$$\frac{\cancel{9}(x-1)^2}{\cancel{144}\,16} - \frac{\cancel{16}(y-3)^2}{\cancel{144}\,9} = \frac{\cancel{144}}{\cancel{144}}\ 1$$

Simplify

$$\boxed{\frac{(x-1)^2}{16} - \frac{(y-3)^2}{9} = 1}$$

Standard form.

This is the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

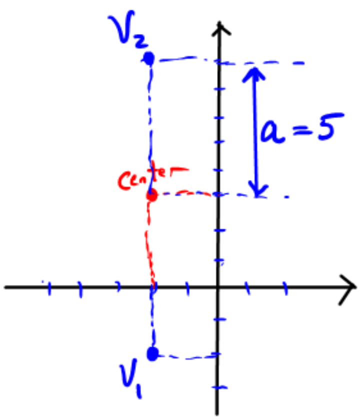
shifted 1 unit right, 3 units up.

Keep in mind : Center, Vertices, Foci are on the same line, always.

Example 4: Write an equation of the hyperbola with center at (-2, 3), one vertex is at (-2, -2) V_1 and eccentricity is 2.

Center gives the shiftment of hyperbola

A quick sketch of problem:



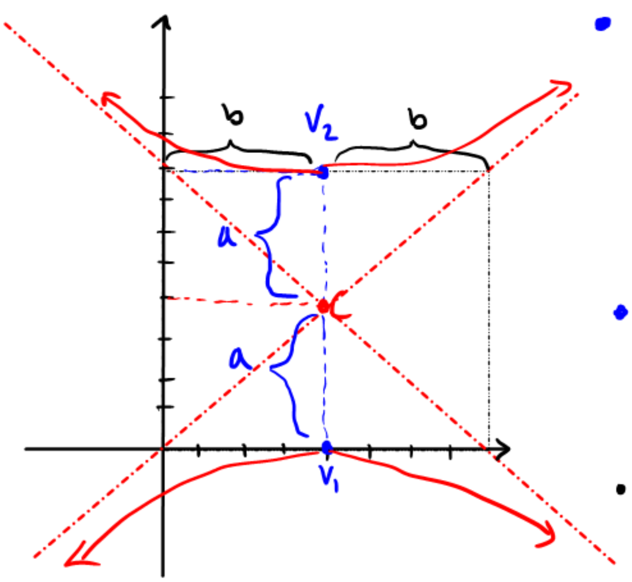
- By picture, it's a vertical hyperbola centered @ (-2,3)
- Vertices are symmetric w.r.t. center, hence $V_2(-2,8)$, and $\underline{a=5}$. $\Rightarrow a^2=25$
- Need b, $e=\frac{c}{a} \Rightarrow c=e\cdot a=2\cdot 5=10$

$c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2=10^2-5^2=75$

$$\Rightarrow \boxed{\frac{(y-3)^2}{25}-\frac{(x+2)^2}{75}=1}$$

Exercise

Example 5: Write an equation of the hyperbola if the vertices are $\underbrace{(4, 0)}_{V_1}$ and $\underbrace{(4, 8)}_{V_2}$ and the asymptotes have slopes ± 1.



- Vertices lined vertically $\Rightarrow$ Vertical hyperbola.

Center is the midpoint of transverse axis.

$\Rightarrow \underbrace{C=(4,4)}$ shifts of hyperbola.

- $a=$ half of transverse length $= 8/2 = 4$.

$\underline{a^2=16}$.

- Being vertical, slant asymptotes are $y=\pm\frac{a}{b}x$

$\Rightarrow \frac{a}{b}=1 \Rightarrow a=b \Rightarrow b^2=16$

$$\Rightarrow \boxed{\frac{(y-4)^2}{16}-\frac{(x-4)^2}{16}=1}$$