Popper 06

1. A
2. A
3. A
4. A
5. A

Math 1330 - Chapter 8
Systems: Identify Equations, Point of Intersection of Equations

Classification of Second Degree Equations – We'll look at the general equation

When you write a conic section in its general form, you have an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. (All of the equations we have seen so far have a value for B that is 0.)

No xy showed up in our examples. B=0. What if B≠0?

We graphed the following examples in the past sections:

$5x^2 + 5y^2 - 20x + 10y = 20$ (a circle) ⟺ both positive square terms, and same coefficients.

$y^2 - 6y = 8x + 7$ (a parabola) ⟺ just one square term

$4x^2 - 8x + 9y^2 - 54y = -49$ (an ellipse) ⟺ both positive square terms, different coefficients

$9x^2 - 16y^2 - 18x + 96y = 279$ (a hyperbola) ⟺ square terms of opposite signs.

With only minimal work, you can determine if an equation in this form is a circle, an ellipse, a parabola or a hyperbola.

Identify each conic section from its equation:

a) $12x = y^2$

parabola – horizontal

b) $\frac{(x-2)^2}{9} - \frac{(y+2)^2}{16} = 1$

hyperbola – horizontal

c) $\frac{(x+4)^2}{4} + \frac{(y-1)^2}{9} = 1$ different

ellipse – vertical

d) $\frac{(x+4)^2}{4} + \frac{(y-1)^2}{4} = 1$ same – circle

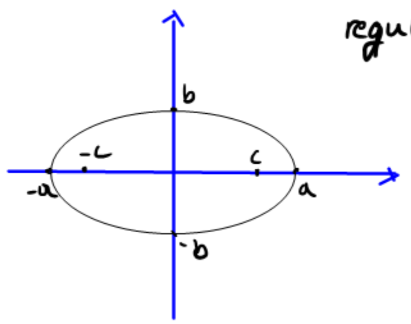
- General Equation for a Conic Section is

$$Ax^2 + \underline{B}xy + Cy^2 + Dx + Ey + F = 0$$

- Up to now, we have dealt with conic sections for which $B=0$. According to their definitions we built them.

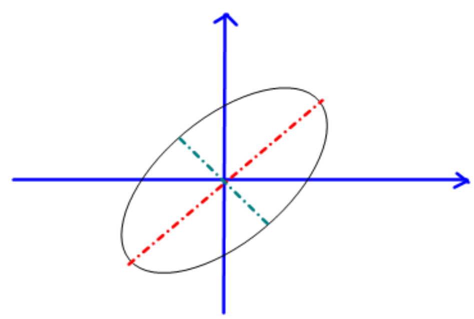
If $B \neq 0$, then the conic sections get "rotated" in the plane. What does that mean?

$B=0$



regular ellipse

$B \neq 0$



still ellipse

- Therefore the values A, B, C are crucial into determining the type of the conic section you are working with.

$\Rightarrow$ It is proved that, no matter how you rotate the graph, a <u>valid</u> conic section can be identified by values of A, B, C.

The general equation <u>stands</u> for one of the conic section we have learned.
If not, it's called <u>degenerate conic section</u>.

<u>ex</u>. $(x-1)^2 + (y+2)^2 = 3$
VALID

$(x-1)^2 + (y+2)^2 = 0$
DEGENERATE.

Classification of Second Degree Equations

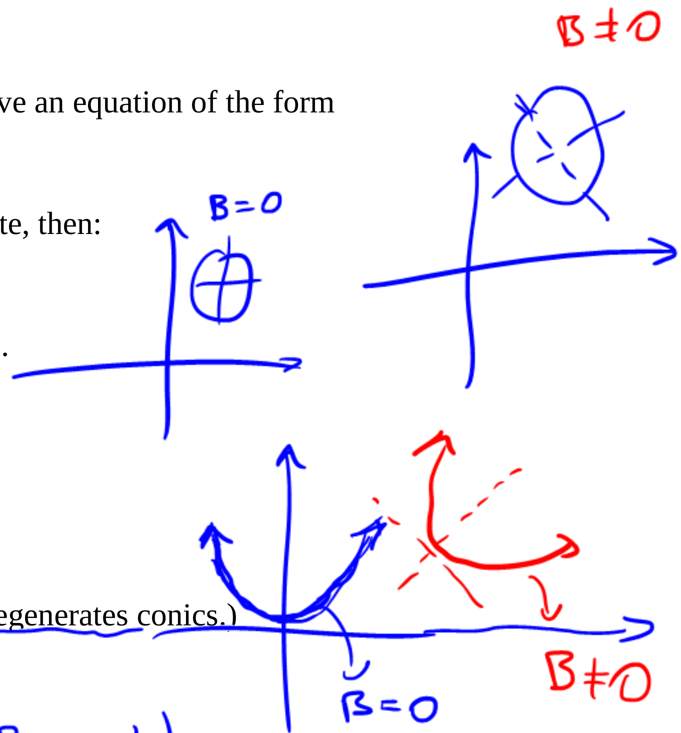
When you write a conic section in its general form, you have an equation of the form

$\Rightarrow$ $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$:

If A, B and C are not all 0, and if the graph is not degenerate, then:

important

- The graph is a **circle** if $B^2 - 4AC < 0$ and $A = C$.
- The graph is an **ellipse** if $B^2 - 4AC < 0$ and $A \neq C$.
- The graph is a **parabola** if $B^2 - 4AC = 0$.
- The graph is a **hyperbola** if $B^2 - 4AC > 0$.



Remember, if there is no "xy" term, then B = 0.

Example: Identify each conic. (Note, none of these are degenerates conics.)

a. $6x^2 - 4xy + 3y^2 + 5x - 7y + 3 = 0$

$6 \neq 3 \Rightarrow$ ellipse - rotated $(B = -4)$

$A \neq C$, $B^2 - 4AC = (-4)^2 - 4 \cdot 6 \cdot 3 < 0$

b. $2x^2 - 8y^2 - 6x - 16y - 25 = 0$

hyperbola - horizontal $(B=0)$ $\quad B^2 - 4AC = 0 - 4 \cdot 2(-8) = 64 > 0$

c. $-3x^2 + 5x - 12y - 7 = 0$

one square, parabola - vertical $(B=0)$

$B^2 - 4AC = 0^2 - 0 = 0$

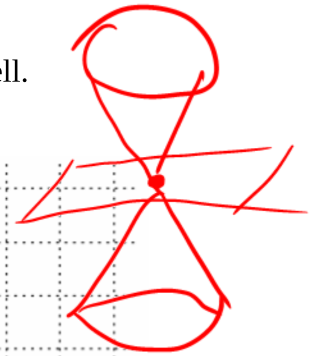
d. $4x^2 + 4y^2 - 24x - 16y - 72 = 0$

match

circle $(B=0)$ $\quad B^2 - 4AC = 0^2 - 4 \cdot 4 \cdot 4 = -64$

$A = C$.

Some degenerate conic sections



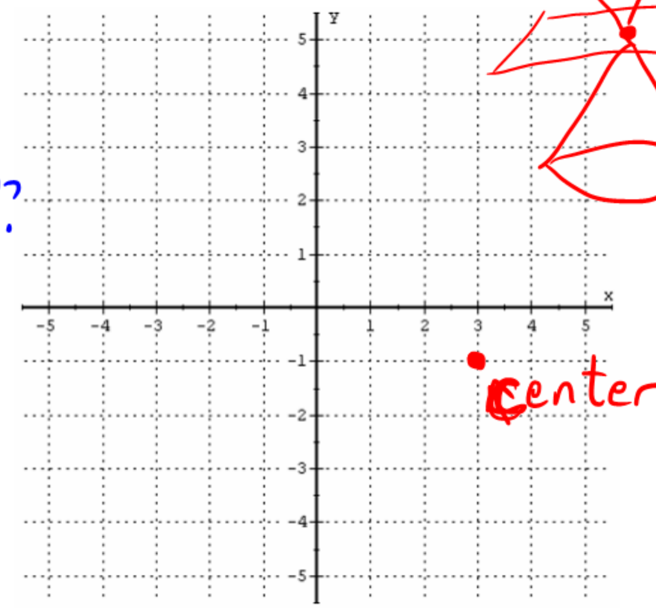
Sometimes equations that look like they should be conic sections do not behave very well.

Example 1: Graph $\underbrace{(x-3)^2+(y+1)^2}=0$

(just a point)

looks like a circle

but radius $=0$???



$$\underbrace{(\quad)^2+(\quad)^2}_{\text{never negative}}=0$$

$\Rightarrow x-3=0\,,\quad y+1=0$

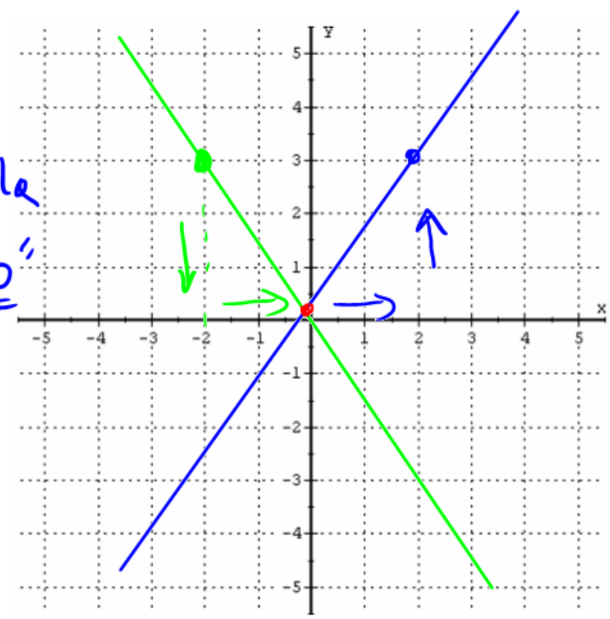
$x=3 \qquad y=-1$

Example 2: Graph $\underbrace{9x^2-4y^2}=0$?

(2 lines)

looks like a hyperbola

but it shouldn't be "0"



$$9x^2-\cancel{4y^2}=0$$

$$\cancel{+4y^2} \qquad +4y^2$$

$$9x^2=4y^2$$

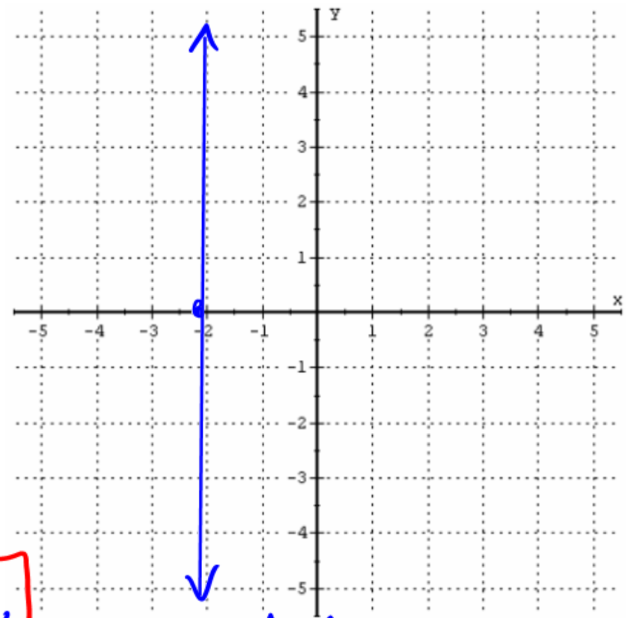
$$\Rightarrow y^2=\frac{9}{4}x^2$$

$$\Rightarrow y=\pm\sqrt{\frac{9}{4}x^2}=\pm\frac{3}{2}x$$

$$\Rightarrow \boxed{y=\frac{3}{2}x\,,\ y=-\frac{3}{2}x}$$

2 lines

Example 3: Graph $2y+(x+2)^2 = 2y$



looks like a parabola

but

$$\cancel{2y} + (x+2)^2 = \cancel{2y}$$
$$-\cancel{2y} \qquad\qquad -\cancel{2y}$$

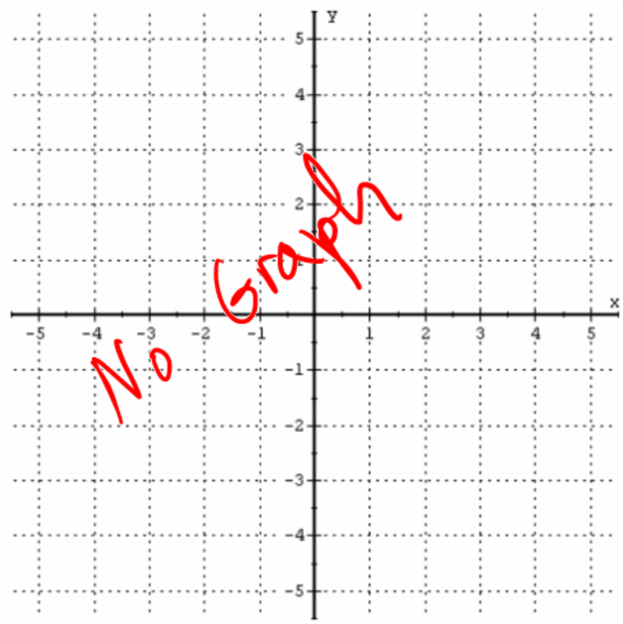
$(x+2)^2 = 0$, No

$\Rightarrow x+2=0 \Rightarrow \boxed{x=-2}$

vertical line

Example 4: Graph $2x^2+3y^2=-1$

(no graph)



looks like an ellipse,

but

$$\underbrace{2x^2+3y^2}_{\text{never negative}} = -1$$

No solution

These are all examples of ***degenerate conic sections*.** Instead of getting the graphs you expect, you have a point (Example 1), two lines (Example 2) and a single line (Example 3) and no graph at all (Example 4). You will not see these very often, but you should be aware of them.

extended example 3

a) $\cancel{2y} + (x+2)^2 = \cancel{2y} + 9$, not a parabola

$-\cancel{2y} \qquad\qquad\qquad\qquad -\cancel{2y}$

$$(x-2)^2 = 9$$

$\Rightarrow x+2=3$ or $x+2=-3$

$\boxed{x=1}$ or $\boxed{x=-5}$

2 vertical lines

b) $\cancel{2y} + (x+2)^2 = \cancel{2y} - 4$ not a parabola

$-\cancel{2y} \qquad\qquad\qquad\qquad -\cancel{2y}$

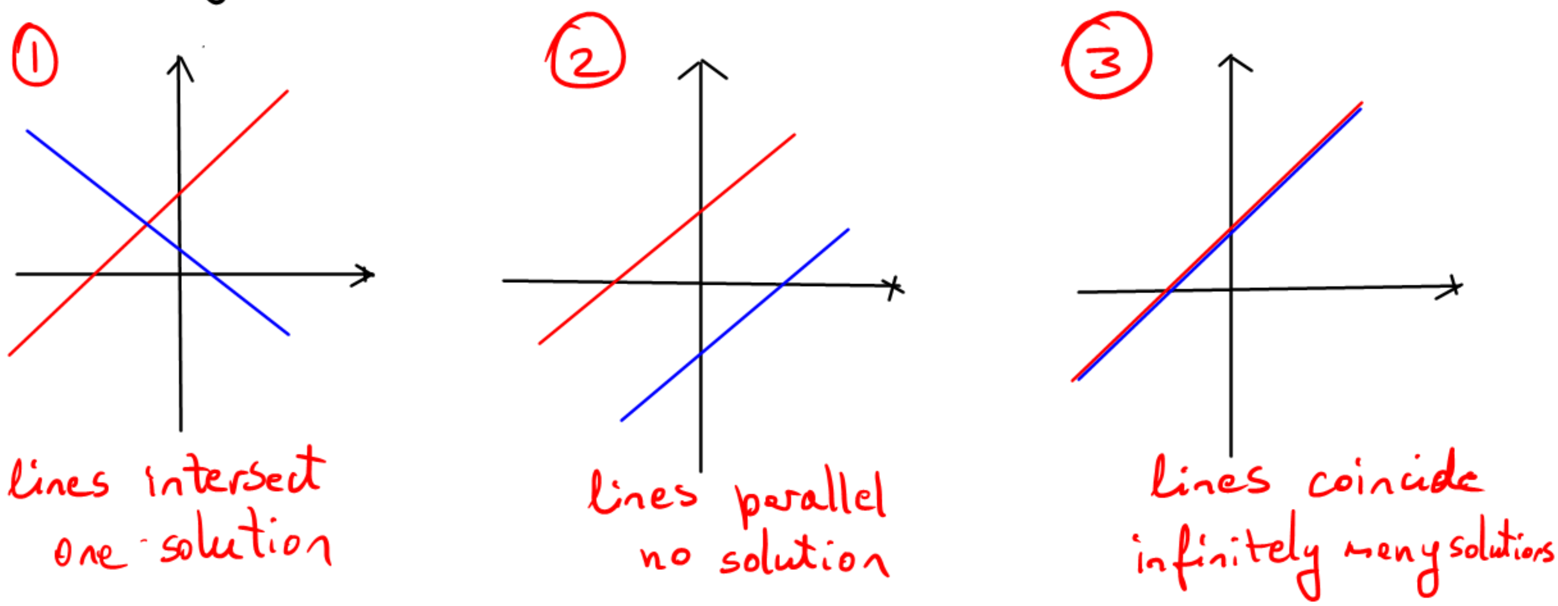
$$(x+2)^2 = -4$$

impossible, No Graph.

From College Algebra, we have seen a linear system to be

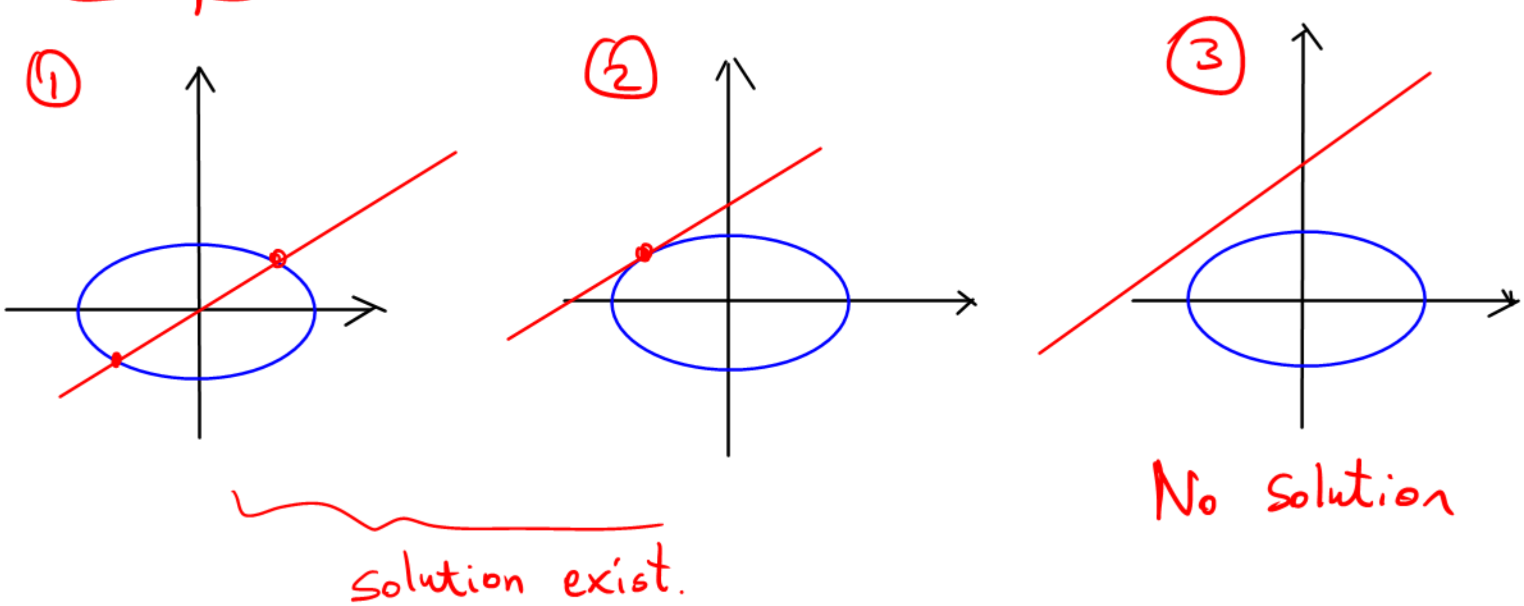
$$* \begin{cases} 2x+y=3 \\ y-4x=5 \end{cases}$$

There are three possible situations.



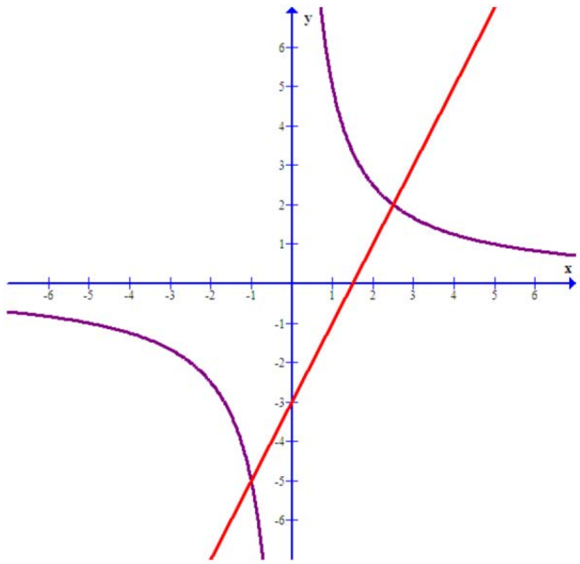
Here, we'll see interactions among conic sections, lines and other graphs.

example



Systems of Second Degree Equations

When we graph two conic sections or a conic section and a line on the same coordinate plane, their graphs may contain points of intersection. The graph below shows a hyperbola and a line and contains two points of intersection.



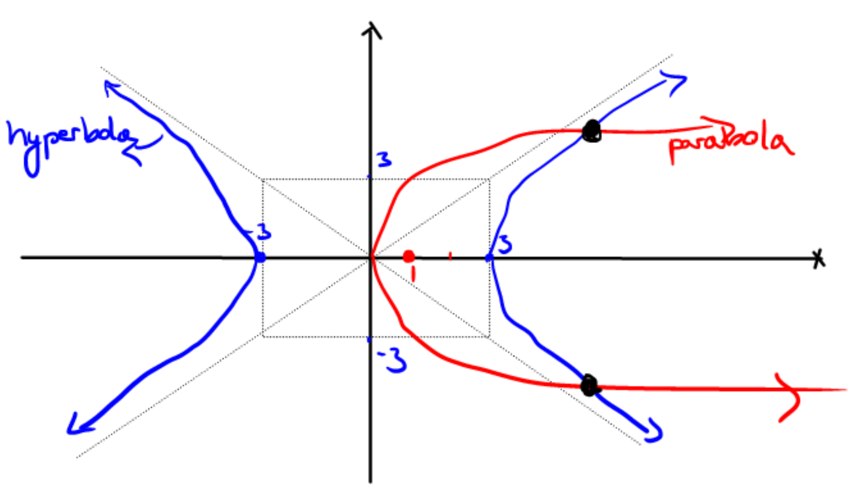
We want to be able to find the points of intersection. To do this, we will solve a system of equations, but now one or both of the equations will be second degree equations. Determining the points of intersection graphically is difficult, so we will do these algebraically.

Example 5: Determine the number of points of intersection for the system.

For questions like this, let's put graph:

$$\frac{x^2}{9} - \frac{y^2}{9} = 1 \longrightarrow \text{horizontal hyperbola},\ a = 3$$

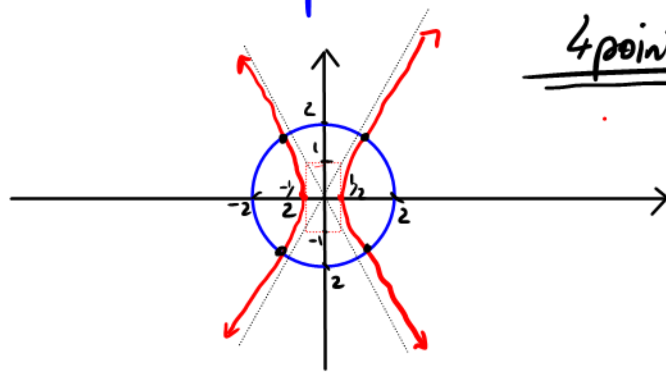
$$y^2 = 4x \longrightarrow \text{horizontal parabola},\ p = 1$$



2 points of intersection

Example 6: Solve the system of equations: Find the points!

$$+\begin{cases} ① \; x^2 + \cancel{y^2} = 4 & \rightarrow \text{circle} \\ ② \; 4x^2 - \cancel{y^2} = 1 & \rightarrow \text{hyperbola horizontal} \end{cases}$$



4 points

$$\frac{\cancel{5}x^2}{\cancel{5}} = \frac{\cancel{5}}{\cancel{5}}$$

$$x^2 = 1 \implies x = \pm 1$$

Now pick $x^2+y^2=4$.

2 points $\begin{cases} x = 1 \\ x^2 + y^2 = 4 \end{cases} \implies 1^2 + y^2 = 4 \implies y^2 = 3 \implies y = \pm\sqrt{3}$

2 points $\begin{cases} x = -1 \\ x^2 + y^2 = 4 \end{cases} \implies (-1)^2 + y^2 = 4 \implies y^2 = 3 \implies y = \pm\sqrt{3}$

$$\implies \boxed{(1,\sqrt{3}),\ (1,-\sqrt{3}),\ (-1,\sqrt{3}),\ (-1,-\sqrt{3})}$$

Work on your own through the rest of exercises.

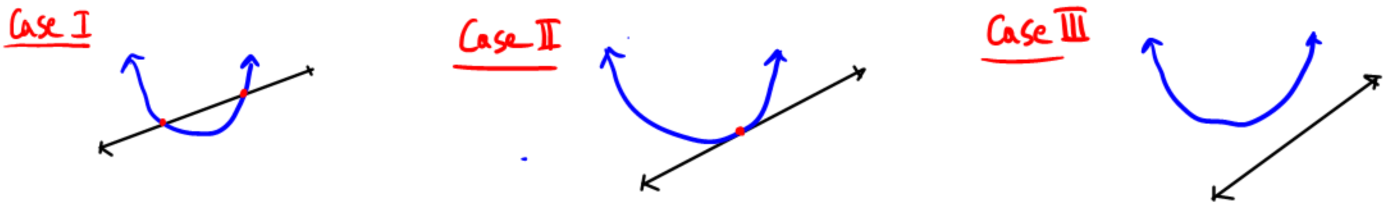
exercise

Example 7: Solve the system of equations:

$f(x) = x^2 - 4x + 11$ ← quadratic function, upward vertical parabola

$g(x) = 5x - 3$ ← linear function, increasing line

Some guessing:



$$\begin{cases} y_1 = x^2 - 4x + 11 \\ y_2 = 5x - 3 \end{cases} \iff \text{They have common solutions if } y_1 = y_2$$

$$x^2 \underset{-5x}{-4x} \underset{+3}{+11} = \underset{-5x}{5x} \underset{+3}{-3}$$

$$x^2 - 9x + 14 = 0$$

$$(x-2)(x-7) = 0$$

$$\Rightarrow x = 2 \quad \text{or} \quad x = 7$$

$$\begin{cases} x = 2 \\ y = 5x - 3 = 5 \cdot 2 - 3 = 7 \end{cases} \Rightarrow (2, 7)$$

$$\begin{cases} x = 7 \\ y = 5x - 3 = 5 \cdot 7 - 3 = 32 \end{cases} \Rightarrow (7, 32)$$

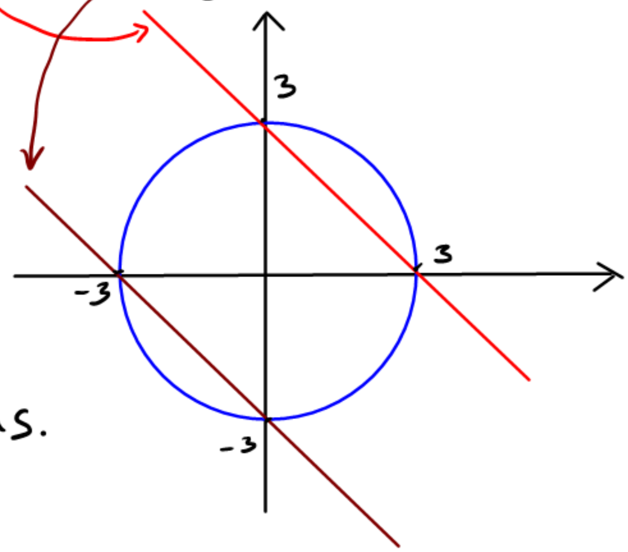
$\boxed{\text{Solution Points } (2,7), (7,32)}$ Case I exercise

Example 8: Solve the system of equations:

$x^2 + y^2 = 9$ ← cirle at origin, radius = 3

$(x+y)^2 = 9$ ← two lines, $\boxed{x+y=3}$ or $\boxed{x+y=-3}$

A correct graph of the given equations give the solutions.



Without graph:

$$\begin{cases} x^2+y^2=9 & (1) \\ (x+y)^2=9 & (2) \end{cases} \Rightarrow \quad x^2+y^2 = (x+y)^2 \text{ FOIL}$$

$$\cancel{x^2}+\cancel{y^2} = \cancel{x^2}+2xy+\cancel{y^2}$$

$$\Rightarrow 2xy = 0$$

$x=0$ or $y=0$

$$\begin{cases} x=0 \\ x^2+y^2=9 \Rightarrow 0^2+y^2=9 \Rightarrow y = \pm\sqrt{9} = \pm 3. \end{cases}$$

$$\begin{cases} y=0 \\ x^2+y^2=9 \Rightarrow x^2+0^2=9 \Rightarrow x = \pm\sqrt{9} = \pm 3 \end{cases}$$

Thus, we got four solutions

$$\begin{cases} x=0 \\ y=\pm 3 \end{cases} \quad \text{or} \quad \begin{cases} y=0 \\ x=\pm 3 \end{cases}$$

$$\Rightarrow \boxed{(0,3),\ (0,-3),\ (3,0),\ (-3,0)}$$

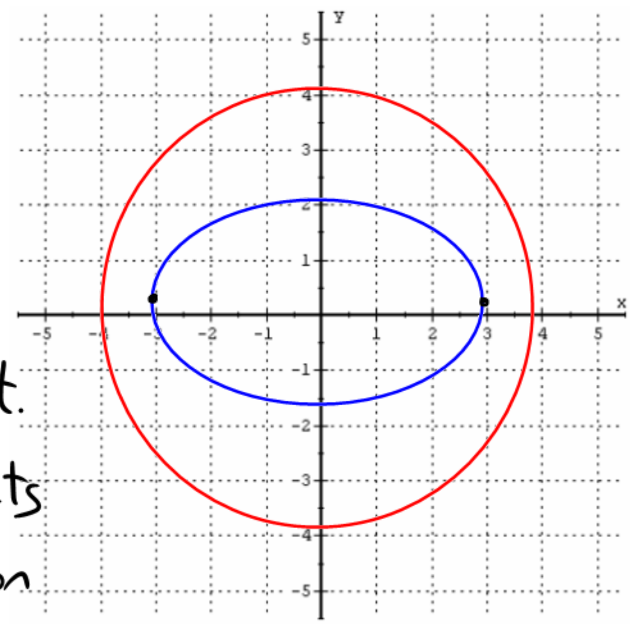
These are same as from the graph.

They should match, always!!!

exercise

Example 9: Graph each and determine the number of points of intersection.

$$* \begin{cases} x^2 + y^2 = 16 & \leftarrow \text{circle}, r=4 \\ \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 & \leftarrow \text{ellipse, horizontal} \quad a=3,\ b=2 \end{cases}$$



Graphically,

it is seen easily they don't intersect.

No intersection points mean no solution to the system (*).

Without graph:

$$\begin{cases} x^2+y^2=16 \\ \dfrac{x^2}{9}+\dfrac{y^2}{4}=1 \end{cases} \underset{\text{(with no fraction)}}{\overset{\text{Rewrite}}{\Longleftrightarrow}} \begin{cases} x^2+y^2=16 & \text{①} \\ 4x^2+9y^2=36 & \text{②} \end{cases}$$

① $x^2+y^2=16 \Rightarrow x^2=16-y^2$

② $4x^2+9y^2=36 \Rightarrow 4\cdot(16-y^2)+9y^2=36$

$64-4y^2+9y^2=36 \Rightarrow \underbrace{5y^2}_{\text{positive}} + \underbrace{-28}_{\text{negative}}$

No Solution. $\square$