Section 6.2 – Double and Half Angle Formulas

Now suppose we are interested in finding $\sin(2A)$. We can use the sum formula for sine to develop this identity:

$$\sin(2A) = \sin(A + A)$$
$$= \sin A \cos A + \sin A \cos A$$
$$= 2 \sin A \cos A$$

Similarly, we can develop a formula for $\cos(2A)$:

$$\cos(2A) = \cos(A + A)$$
$$= \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A$$

We can restate this formula in terms of sine only or in terms of cosine only by using the Pythagorean theorem and making a substitution. So we have:

$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$= 1 - 2 \sin^2 A$$
$$= 2 \cos^2 A - 1$$

We can also develop a formula for $\tan(2A)$:

$$\tan(2A) = \tan(A + A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$
$$= \frac{2 \tan A}{1 - \tan^2 A}$$

These three formulas are called the double angle formulas for sine, cosine and tangent.

Besides these formulas, we also have the so-called half-angle formulas for sine, cosine and tangent, which are derived by using the double angle formulas for sine, cosine and tangent, respectively.
Double – Angle Formulas

\[ \sin(2A) = 2\sin A \cos A \]
\[ \cos(2A) = \cos^2 A - \sin^2 A \]
\[ = 2\cos^2 A - 1 \]
\[ = 1 - 2\sin^2 A \]
\[ \tan(2A) = \frac{2\tan A}{1 - \tan^2 A} \]

Half – Angle Formulas

\[ \sin \left( \frac{A}{2} \right) = \pm \sqrt{\frac{1 - \cos A}{2}} \]
\[ \cos \left( \frac{A}{2} \right) = \pm \sqrt{\frac{1 + \cos A}{2}} \]
\[ \tan \left( \frac{A}{2} \right) = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A} \]

Note: In the half-angle formulas the ± symbol is intended to mean either positive or negative but not both, and the sign before the radical is determined by the quadrant in which the angle \( \frac{A}{2} \) terminates.
Example 1: Suppose that $\cos \alpha = -\frac{4}{7}$ and $\frac{\pi}{2} < \alpha < \pi$. Find

a. $\cos(2\alpha)$

b. $\sin(2\alpha)$

c. $\tan(2\alpha)$

Example 2: Simplify each:

a. $2 \sin 45^\circ \cos 45^\circ$
b. \( \cos^2 \frac{\pi}{9} - \sin^2 \frac{\pi}{9} \)

c. \( \frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ} \)

d. \( 1 - 2 \sin^2(6A) \)

Now we’ll look at the kinds of problems we can solve using half-angle formulas.
Example 3: Use a half-angle formula to find the exact value of each.

a. $\sin 15^\circ$

b. $\cos \left( \frac{5\pi}{8} \right)$

c. $\tan \left( \frac{7\pi}{12} \right)$
Example 4: Answer these questions for \( \cos \theta = \frac{4}{9}, \frac{3\pi}{2} < \theta < 2\pi \).

a. In which quadrant does the terminal side of the angle lie?

b. Complete the following: \( \_\_ < \frac{\theta}{2} < \_\_ \)

c. In which quadrant does the terminal side of \( \frac{\theta}{2} \) lie?

d. Determine the sign of \( \sin \left( \frac{\theta}{2} \right) \).

e. Determine the sign of \( \cos \left( \frac{\theta}{2} \right) \).

f. Find the exact value of \( \sin \left( \frac{\theta}{2} \right) \).

g. Find the exact value of \( \cos \left( \frac{\theta}{2} \right) \).

h. Find the exact value of \( \tan \left( \frac{\theta}{2} \right) \).