Math 1330 – Section 8.2
Ellipses

**Definition:** An ellipse is the set of all points, the sum of whose distances from two fixed points is constant. Each fixed point is called a focus (plural = foci).

**Basic ellipses (centered at origin):**

**Basic “vertical” ellipse:**

Equation: \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \ a > b \)

Foci: \((0, \pm c)\), where \(c^2 = a^2 - b^2\)

Vertices: \((0, \pm a)\)

Eccentricity: \(e = \frac{c}{a}\)

**Basic “horizontal” ellipse:**

Equation: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a > b \)

Foci: \((\pm c, 0)\), where \(c^2 = a^2 - b^2\)

Vertices: \((\pm a, 0)\)

Eccentricity: \(e = \frac{c}{a}\)

The eccentricity provides a measure on how much the ellipse deviates from being a circle. The eccentricity \(e\) is a number between 0 and 1.

- small \(e\): graph resembles a circle (foci close together)
- large \(e\): flatter, more elongated (foci far apart)
- if the foci are the same, it’s a circle!
Graphing ellipses:  \[ \Rightarrow \text{Bring it in standard form} \]

To graph an ellipse with center at the origin:

- Rearrange into the form \( \frac{x^2}{\text{number}} + \frac{y^2}{\text{number}} = 1 \).
- Decide if it’s a “horizontal” or “vertical” ellipse.
  - if the bigger number is under \( x^2 \), it’s horizontal (longer in \( x \)-direction).
  - if the bigger number is under \( y^2 \), it’s vertical (longer in \( y \)-direction).
- Use the square root of the number under \( x^2 \) to determine how far to measure in \( x \)-direction.
- Use the square root of the number under \( y^2 \) to determine how far to measure in \( y \)-direction.
- Draw the ellipse with these measurements. Be sure it is smooth with no sharp corners
- \( c^2 = a^2 - b^2 \) where \( a^2 \) and \( b^2 \) are the denominators. So \( c = \sqrt{\text{big denom} - \text{small denom}} \)
- The foci are located \( c \) units from the center on the long axis.

To graph an ellipse with center not at the origin:

- Rearrange (complete the square if necessary) to look like \( \frac{(x-h)^2}{\text{number}} + \frac{(y-k)^2}{\text{number}} = 1 \).
- Start at the center \( (h,k) \) and then graph it as before.

When graphing, you will need to find the orientation, center, values for \( a \), \( b \) and \( c \), vertices, foci, lengths of the major and minor axes and eccentricity.
Example 1: Find all relevant information and graph \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \).

<table>
<thead>
<tr>
<th>Orientation:</th>
<th>horizontal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center:</td>
<td>((0,0))</td>
</tr>
<tr>
<td>Vertices:</td>
<td>((4,0), (-4,0))</td>
</tr>
<tr>
<td>Foci:</td>
<td>((\pm 4,0))</td>
</tr>
<tr>
<td>Length of major axis:</td>
<td>8</td>
</tr>
<tr>
<td>Length of minor axis:</td>
<td>6</td>
</tr>
<tr>
<td>Eccentricity:</td>
<td>(e = \frac{\sqrt{7}}{4} \approx 0.66 )</td>
</tr>
</tbody>
</table>

Example 2: Find all relevant information and graph \( \frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1 \).

<table>
<thead>
<tr>
<th>Orientation:</th>
<th>vertical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center:</td>
<td>((1,-2))</td>
</tr>
<tr>
<td>Vertices:</td>
<td>((1,3), (1,-3))</td>
</tr>
<tr>
<td>Foci:</td>
<td>((1,2), (1,-6))</td>
</tr>
<tr>
<td>Length of major axis:</td>
<td>10</td>
</tr>
<tr>
<td>Length of minor axis:</td>
<td>6</td>
</tr>
<tr>
<td>Eccentricity:</td>
<td>(e = \frac{4}{5} = 0.8 )</td>
</tr>
</tbody>
</table>
Example 3: Write the equation in standard form. Find all relevant information and graph:

\[4x^2 - 8x + 9y^2 - 54y = -49.\]

- Group x-terms together, y-terms together:
  \[(4x^2 - 8x) + (9y^2 - 54y) = -49\]

- Factor coefficients in front of squares:
  \[4(x^2 - 2x + 1) + 9(y^2 - 6y + 9) = -49 + 4\cdot1 + 9\cdot9\]

- Complete the square:
  \[4(x-1)^2 + 9(y-3)^2 = 36\]

- Divide both sides by 36:
  \[\frac{(x-1)^2}{9} + \frac{(y-3)^2}{4} = 1\]

Graph is easy now!

Example 4: Find the equation for the ellipse satisfying the given conditions.

- Foci \((\pm 3, 0)\), vertices \((\pm 5, 0)\)
  - Over x-axis \(a = 5\)
  - Vertical axis \(b = 3\)
  - Center is midpoint of foci \((0, 0)\)

\[\frac{x^2}{25} + \frac{y^2}{16} = 1\]

Example 5: Write an equation of the ellipse with vertices \((5, 9)\) and \((5, 1)\) if one of the foci is \((5, 7)\).

- By the graph, it is vertical and shifted.
  - Center = midpoint of major axis = \((5, 5)\)
  - Foci \((5, 7)\)
  - Length of major axis = \(2a = 8\)
  - \(a = 4\)

\[\Rightarrow \frac{(x-5)^2}{12} + \frac{(y-5)^2}{16} = 1\]