Math 1330 – Section 8.2
Ellipses

Definition: An ellipse is the set of all points, the sum of whose distances from two fixed points is constant. Each fixed point is called a focus (plural = foci).

Basic ellipses (centered at origin):

Electronic grading

Basic “vertical” ellipse:

Equation: \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \ a > b \)

Foci: \((0, \pm c)\), where \( c^2 = a^2 - b^2 \)

Vertices: \((0, \pm a)\)

Eccentricity: \( e = \frac{c}{a} \)

Basic “horizontal” ellipse:

Equation: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a > b \)

Foci: \((\pm c, 0)\), where \( c^2 = a^2 - b^2 \)

Vertices: \((\pm a, 0)\)

Eccentricity: \( e = \frac{c}{a} \)

The eccentricity provides a measure on how much the ellipse deviates from being a circle. The eccentricity \( e \) is a number between 0 and 1.

- small \( e \): graph resembles a circle (foci close together)
- large \( e \): flatter, more elongated (foci far apart)
- if the foci are the same, it’s a circle!
Graphing ellipses:

To graph an ellipse with center at the origin:

- Rearrange into the form \( \frac{x^2}{\text{number}} + \frac{y^2}{\text{number}} = 1 \).

- Decide if it’s a “horizontal” or “vertical” ellipse.
  - if the bigger number is under \( x^2 \), it’s horizontal (longer in \( x \)-direction).
  - if the bigger number is under \( y^2 \), it’s vertical (longer in \( y \)-direction).

- Use the square root of the number under \( x^2 \) to determine how far to measure in \( x \)-direction.

- Use the square root of the number under \( y^2 \) to determine how far to measure in \( y \)-direction.

- Draw the ellipse with these measurements. Be sure it is smooth with no sharp corners.

- \( c^2 = a^2 - b^2 \) where \( a^2 \) and \( b^2 \) are the denominators. So \( c = \sqrt{\text{big denom} - \text{small denom}} \)

- The foci are located \( c \) units from the center on the long axis.

To graph an ellipse with center not at the origin:

- Rearrange (complete the square if necessary) to look like \( \frac{(x-h)^2}{\text{number}} + \frac{(y-k)^2}{\text{number}} = 1 \).

- Start at the center \((h,k)\) and then graph it as before.

When graphing, you will need to find the orientation, center, values for \( a, b \) and \( c \), vertices, foci, lengths of the major and minor axes and eccentricity.
Example 1: Find all relevant information and graph \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \).

- **Orientation**: horizontal
- **Center**: \((0,0)\)
- **Vertices**: \((4,0), (-4,0)\)
- **Foci**: \((\sqrt{7},0), (-\sqrt{7},0)\)
- **Length of major axis**: \(2a = 2 \cdot 4 = 8\)
- **Length of minor axis**: \(2b = 2 \cdot 3 = 6\)
- **Coordinates of the major axis**: \((4,0), (-4,0)\)
- **Coordinates of the minor axis**: \((0,3), (0,-3)\)
- **Eccentricity**: \(e = \frac{c}{a} = \frac{\sqrt{7}}{4} \approx 0.66\)

Example 2: Find all relevant information and graph \( \frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1 \).

- **Orientation**: vertical
- **Center**: \((1, -2)\)
- **Vertices**: \((1,3), (1, -7)\)
- **Foci**: \((1,2), (1, -6)\)
- **Length of major axis**: \(2a = 10\)
- **Length of minor axis**: \(2b = 6\)
- **Eccentricity**: \(e = \frac{c}{a} = \frac{4}{5} \approx 0.8\)
Example 3: Write the equation in standard form. Find all relevant information and graph:

\[4x^2 - 8x + 9y^2 - 54y = -49.\]

\[
\begin{align*}
4(x^2 - 2x + 1) + 9(y^2 - 6y + 9) &= -49 + 4 + 9 \\
(x-1)^2 + (y-3)^2 &= 36
\end{align*}
\]

\[
\frac{(x-1)^2}{9} + \frac{(y-3)^2}{4} = 1
\]

\[\text{Graph is easy now!}\]

Example 4: Find the equation for the ellipse satisfying the given conditions.

Foci (±3,0), vertices (± 5,0) \[\Rightarrow \text{horizontal}\]

\[
C=5 \quad \text{over x-axis} \quad a=5
\]

\[
\begin{align*}
&\Rightarrow \text{horizontal} \\
&\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\end{align*}
\]

\[
\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1
\]

Example 5: Write an equation of the ellipse with vertices (5, 9) and (5, 1) if one of the foci is (5, 7).

By the graph, it is vertical and shifted.

Center = midpoint of major axis = (5, 5) \[\text{shifted}\]

\[
\text{Foci (5,7) } \Rightarrow C=2 ,
\]

\[
\text{length of major axis } = 2a = 8 \Rightarrow a = 4
\]

\[
\Rightarrow b^2 = a^2 - c^2 = 4^2 - 2^2 = 12
\]

\[
\Rightarrow \frac{(x-5)^2}{12} + \frac{(y-5)^2}{4} = 1
\]